Study Material on CC-8

Department of Mathematics, P. R. Thakur Govt. College
MTMACOR08T: (Semester - 4)

Syllabus:

Unit -1 : Riemann integration: inequalities of upper and lower sums, Darboux integration, Darboux theorem, Riemann conditions of integrability, Riemann sum and definition of Riemann integral through Riemann sums, equivalence of two Definitions. Riemann integrability of monotone and continuous functions, Properties of the Riemann integral; definition and integrability of piecewise continuous and monotone functions. Intermediate Value theorem for Integrals, Fundamental theorem of Integral Calculus.

Unit-2 : Improper integrals, Convergence of Beta and Gamma functions.

Unit-3 : Pointwise and uniform convergence of sequence of functions. Theorems on continuity, derivability and integrability of the limit function of a sequence of functions. Series of functions, Theorems on the continuity and derivability of the sum function of a series of functions; Cauchy criterion for uniform convergence and Weierstrass M-Test.

Unit 4: Fourier series: Definition of Fourier coefficients and series, Reimann Lebesgue lemma, Bessel's inequality, Parseval's identity, Dirichlet's condition. Examples of Fourier expansions and summation results for series.

Unit – 5: Power series, radius of convergence, Cauchy Hadamard Theorem. Differentiation and integration of power series; Abel's Theorem; Weierstrass Approximation Theorem.

1 Riemann Integration

1.1 Partition

Definition. 1.1 Let $[a, b]$ be a closed interval in $\mathbb{R}$. By a *partition* P of $[a, b]$ we mean a finite set of numbers $\{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

We write this partition as $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ or as $P = \{x_0, x_1, x_2, \ldots < x_n\}$, where $a = x_0$, $b = x_n$.

Example. 1.2 Let us consider the interval [0, 1]. Then the set $\{0, .25, .5, .75, 1\}$ is a partition of [0, 1]. Another example of partition of the same interval is $\{0, .1, .45, .6, .8, 1\}$. Note that in the first example the points are equally spaced whereas in the second one the points are in unequal spacing.

Definition. 1.3 Let $[a, b]$ be an interval in $\mathbb{R}$, $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and $Q : a = y_0 < y_1 < y_2 < \cdots < y_m = b$ be two partitions of $[a, b]$. Then P is said to be a *refinement* of Q if $\{x_0, x_1, x_0, \ldots, x_n\} \supset \{y_0, y_1, y_2, \ldots, y_m\}$. It is obvious in this case that $n \geq m$.

EXAMPLE. 1.4 1. Let us consider the partitions $P : 0 < .25 < .5 < .75 < 1$ and $Q : 0 < .2 < .25 < .45 < .5 < .6 < .75 < .9 < 1$ of the interval $[0, 1]$. Then Q is a refinement of P.

2. Let us consider the partitions $P = \{0, .25, .5, .75, 1\}$ and $Q = \{0, .2, .4, .6, .8, 1\}$. Then neither P is a refinement of Q nor Q is a refinement of P.

NOTATION. 1.5 *The set of all partitions of an interval* $[a, b]$ *will be denoted by* $\mathcal{P}[a, b]$. *For* $P_1, P_2 \in \mathcal{P}[a, b]$, *if* P_2 *is a refinement of* P_1, *it will be denoted by* $P_1 \prec P_2$.

THEOREM. 1.6 *Let* $[a, b]$ *be an interval in* $\mathbb{R}$. *Then*

1. *For every* P *in* $\mathcal{P}[a, b]$, $P \prec P$.
2. *If* $P_1, P_2, P_3 \in \mathcal{P}[a, b]$ *such that* $P_1 \prec P_2$ *and* $P_2 \prec P_3$, *then* $P_1 \prec P_3$.
3. *If* $P_1, P_2 \in \mathcal{P}[a, b]$ *then there exists* P_3 *in* $\mathcal{P}[a, b]$ *such that* $P_1 \prec P_3$ *and* $P_2 \prec P_3$.

EXAMPLE. 1.7 Consider the example 2 of 1.4. The partition $R = \{0, 0.2, 0.25, 0.4, 0.5, 0.6,\text{-}0.75, 0.8, 1\}$ is a refinement of both P and Q.

DEFINITION. 1.8 Let $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of the interval $[a, b]$. Then the *norm* of P is defined as $\max\{x_r - x_{r-1} : 1 \leq r \leq n\}$, and is denoted by $||P||$.

EXAMPLE. 1.9 In the example 1.2 norm of the first partition is 0.25 whereas the norm of the second partition is 0.35.

NOTATION. 1.10 *It can be noted that if* P_1, P_2 *are two partitions such that* $P_1 \prec P_2$, *then* $||P_1|| > ||P_2||$.

1.2 Upper and Lower Sum

DEFINITION. 1.11 Let $[a, b]$ be an interval in $\mathbb{R}$, $f : [a, b] \to \mathbb{R}$ be a bounded function, $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a, b]$. For $r = 1, 2, \ldots, n$ define

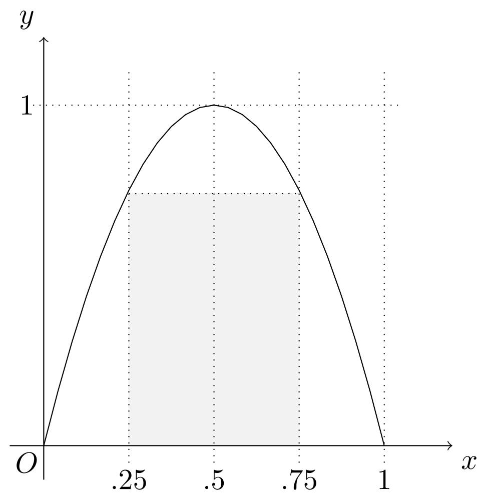
$$M_r = \sup\{f(x) : x_{r-1} \leq x \leq x_r\},\ m_r = \inf\{f(x) : x_{r-1} \leq x \leq x_r\},\ \delta_r = x_r - x_{r-1}.$$

Then the sum $\sum_{r=1}^{n} M_r \delta_r$ is called the *upper sum* of f for the partition P and is denoted bu $U(f, P)$. The sum $\sum_{r=1}^{n} m_r \delta_r$ is called the *lower sum* of f for the partition P and is denoted bu $L(f, P)$.

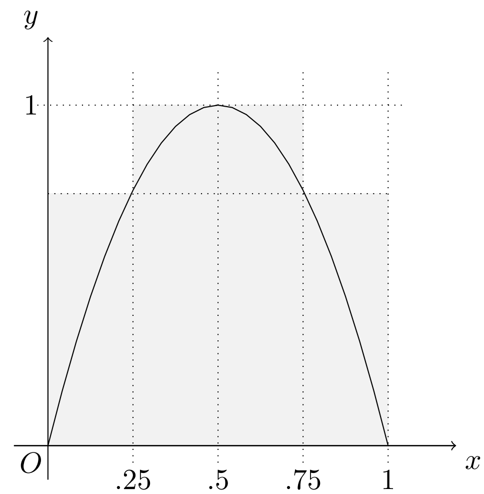
Henceforth by a function we always mean a bounded function.

EXAMPLE. 1.12 1. Let $f(x) = x, 0 \leq x \leq 1$, P be defined as $P = \{0, .25, .5, .75, 1\}$. Then $\delta_1 = \delta_2 = \delta_3 = \delta_4 = .25$, $M_1 = \sup\{x : 0 \leq x \leq .25\} = .25, m_1 = \inf\{x : 0 \leq x \leq .25\} = 0, M_2 = \sup\{x : .25 \leq x \leq .5\} = .5, m_2 = \inf\{x : .25 \leq x \leq .5\} = .25, M_3 = \sup\{x : .5 \leq x \leq .75\} = .75, m_3 = \inf\{x : .5 \leq x \leq .75\} = .5, M_4 = \sup\{x : .75 \leq x \leq 1\} = 1, m_4 = \inf\{x : .75 \leq x \leq 1\} = .75$. So $U(f, P) = M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + M_4\delta_4 = 0.0625 + 0.125 + 0.1875 + 0.25 = 0.625$. Similarly $L(f, P) = 0.375$

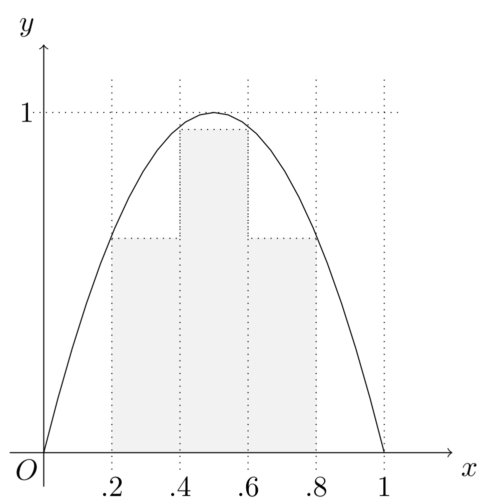
2. Let $f(x) = 4x(1 - x), 0 \leq x \leq 1$. $P_1 = \{0, .25, .5, .75, 1\}$ and $P_2 = \{0, .2, .4, .6, .8, 1\}$. Then it can easily be verified that $U(f, P_1) = 0.875, L(f, P_1) = 0.375, U(f, P_2) = 0.84, L(f, P_2) = .448$.



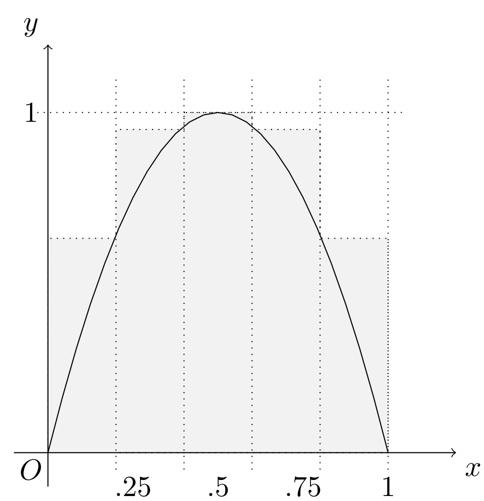
$L(f, P_1)$



$U(f, P_1)$



$L(f, P_2)$



$U(f, P_2)$

THEOREM. 1.13 *For a function f defined on an interval $[a, b]$ and for any partition P of $[a, b]$, $L(f, P) \leq U(f, P)$.*

Proof. This follows immediately since $m_r \leq M_r$ for every $r, 1 \leq r \leq n$.

Theorem. 1.14 *Let f be a function defined on an interval $[a,b]$, P, Q be two partitions of $[a,b]$ such that Q is a refinement of P. Then $U(f,P) \geq U(f,Q)$ and $L(f,P) \leq L(f,Q)$.*

Proof. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$. Note that Q is obtained by inserting finite number of points between the elements of P. It is sufficient to show that by inserting a single point the result holds. Let us consider the partition $P_1 : a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < y < x_i < \cdots < x_n = b$ which is obtained by inserting a single point y between x_{i-1} and x_i. Let

$$M_r = \sup\{f(x) : x_{r-1} \leq x \leq x_r\},\ m_r = \inf\{f(x) : x_{r-1} \leq x \leq x_r\},\ \ 1 \leq r \leq n.$$

Also let,

$$M_i' = \sup\{f(x) : x_{i-1} \leq x \leq y\},\ m_i' = \inf\{f(x) : x_{i-1} \leq x \leq y\}.$$

and

$$M_i'' = \sup\{f(x) : y \leq x \leq x_i\},\ m_i'' = \inf\{f(x) : y \leq x \leq x_i\}.$$

Then $M_i \geq M_i',\ M_i \geq M_i''$ and $m_i \leq m_i',\ m_i \leq m_i''$.

Now

$$\begin{aligned}
U(f,P) &= \sum_{r=1}^{n} M_r(x_r - x_{r-1}) \\
&= \sum_{r=1}^{i-1} M_r(x_r - x_{r-1}) + M_i(x_i - x_{i-1}) + \sum_{r=i+1}^{n} M_r(x_r - x_{r-1}) \\
&= \sum_{r=1}^{i-1} M_r(x_r - x_{r-1}) + M_i(x_i - y) + M_i(y - x_{i-1}) + \sum_{r=i+1}^{n} M_r(x_r - x_{r-1}) \\
&\geq \sum_{r=1}^{i-1} M_r(x_r - x_{r-1}) + M_i'(x_i - y) + M_i''(y - x_{i-1}) + \sum_{r=i+1}^{n} M_r(x_r - x_{r-1}) \\
&= U(f,P_1).
\end{aligned}$$

And

$$\begin{aligned}
L(f,P) &= \sum_{r=1}^{n} m_r(x_r - x_{r-1}) \\
&= \sum_{r=1}^{i-1} m_r(x_r - x_{r-1}) + m_i(x_i - x_{i-1}) + \sum_{r=i+1}^{n} m_r(x_r - x_{r-1}) \\
&= \sum_{r=1}^{i-1} m_r(x_r - x_{r-1}) + m_i(x_i - y) + m_i(y - x_{i-1}) + \sum_{r=i+1}^{n} m_r(x_r - x_{r-1}) \\
&\leq \sum_{r=1}^{i-1} m_r(x_r - x_{r-1}) + m_i'(x_i - y) + m_i''(y - x_{i-1}) + \sum_{r=i+1}^{n} m_r(x_r - x_{r-1}) \\
&= L(f,P_1).
\end{aligned}$$

Thus $U(f,P) \geq U(f,P_1)$ and $L(f,P) \leq L(f,P_1)$. ■

NOTATION. 1.15 *It is observed that by refining a partition the upper sum is decreased and the lower sum is increased.*

In view of the above two theorems we have the following result.

COROLLARY. 1.16 *Let f be a function defined on an interval $[a,b]$. Then for any two partitions P,Q of $[a,b]$, $U(f,P) \geq L(f,Q)$.*

PROOF. Note that there exists a partition R of $[a,b]$ which is a refinement of both P and Q. Then by the above theorem $U(f,P) \geq U(f,R)$ and $L(f,Q) \leq L(f,R)$. Also by Theorem 1.13 $U(f,R) \geq L(f,R)$. Thus $U(f,P) \geq U(f,R) \geq L(f,R) \geq L(f,Q)$. ■

NOTATION. 1.17 *It is observed that the upper sum for any partition is greater than or equal to the lower sum for any partition. Thus the set $\{U(f,P) : P \in \mathcal{P}[a,b]\}$ is bounded below, any lower sum of f being a lower bound; and the set $\{L(f,P) : P \in \mathcal{P}[a,b]\}$ is bounded above, any upper sum of f being an upper bound.*

DEFINITION. 1.18 The difference $U(f,P) - L(f,P)$ is called the oscillatory sum of f for the partition P of $[a,b]$.

1.3 Definition of Riemann Integration

Let f be a bounded function defined in a closed interval $[a,b]$. It has already been observed that the set of all the upper sums of f is bounded below and the set of all the lower sums of f is bounded above. We define the lower and upper integral as follows:

DEFINITION. 1.19 The infimum of all the upper bounds of f, where the infimum is taken over all the partitions of $[a,b]$, is called the upper integral of f over $[a,b]$ and is denoted by $\int_a^{\overline{b}} f(x)\ dx$. The supremum of all the lower sums of f, where the supremum is taken over all the partitions of $a,b]$, is called the lower integral of f over $[a,b]$ and is denoted by $\int_{\underline{a}}^b f(x)\ dx$. Thus,

$$\int_a^{\overline{b}} f(x)\ dx = \inf \{U(f,P) : P \in \mathcal{P}[a,b]\} \quad \text{and} \quad \int_{\underline{a}}^b f(x)\ dx = \sup \{L(f,P) : P \in \mathcal{P}[a,b]\}.$$

As any lower sum is always less than or equal to any upper sum it immediately follows that $\int_{\underline{a}}^b f(x)\ dx \leq \int_a^{\overline{b}} f(x)\ dx$.

DEFINITION. 1.20 A bounded function f defined on a closed interval $[a,b]$ is said to be *Riemann Integrable*, or simply *R-integrable,* if $\int_a^{\overline{b}} f(x)\ dx = \int_{\underline{a}}^b f(x)\ dx$, the common value is called the Riemann integral of f over $[a,b]$ and is denoted by $\int_a^b f(x)\ dx$.

EXAMPLE. 1.21 1. Every constant function is Rieman integrable. Let $f(x) = k, a \leq x \leq b$. Then for any partition $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$, we have $M_r = m_r = k, 1 \leq r \leq n$, Thus $U(f, P) = L(f, P) = k(b - a)$.

2. The function $f(x) = x,\ 0 \leq x \leq 1$ is R-integrable. For $n \in \mathbb{N}$, consider the partition $P_n : 0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1$, the $r-$th subinterval being $[\frac{r-1}{n}, \frac{r}{n}]$. Since f is increasing in $[0, 1]$, we have $M_r = \sup\{f(x) : \frac{r-1}{n} \leq x \leq \frac{r}{n}\} = \frac{r}{n}$ and $m_r = \inf\{f(x) : \frac{r-1}{n} \leq x \leq \frac{r}{n}\} = \frac{r-1}{n}$. Also $\delta_r = \frac{1}{n}$ for all $r = 1, 2, \ldots, n$. So $U(f, P_n) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \frac{r}{n} \cdot \frac{1}{n} = \frac{1}{n^2}\sum_{r=1}^{n} r = \frac{n(n+1)}{2n^2} = \frac{1}{2}(1 + \frac{1}{n})$ and $L(f, P_n) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} \frac{r-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}\sum_{r=1}^{n}(r - 1) = \frac{(n-1)n}{2n^2} = \frac{1}{2}(1 - \frac{1}{n})$ Taking limit as $n \to \infty$, we have $\lim_{n\to\infty} U(f, P_n) = \frac{1}{2}$ and $\lim_{n\to\infty} L(f, P_n) = \frac{1}{2}$. Note that $\{P_n : n \in \mathbb{N}\} \subset \mathcal{P}[a, b]$. Thus $\frac{1}{2} = \lim_{n\to\infty} U(f, P_n) \geq \int_0^{\bar{1}} f(x)\ dx \geq \int_{\underline{0}}^{1} f(x)\ dx \geq \lim_{n\to\infty} L(f, P_n) = \frac{1}{2}$ Hence $\int_0^{\bar{1}} f(x)\ dx = \int_{\underline{0}}^{1} f(x)\ dx = \frac{1}{2}$. This shows that f is R-integrable in $[0, 1]$ and $\int_0^1 f(x)\ dx = \frac{1}{2}$.

3. Consider the function f in $[a, b]$ defined as follows: $f(x) = 0, x$ irrational, $f(x) = 1, x$ rational. Let $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a, b]$. Since every interval contains rational numbers as well as irrational numbers, we have $M_r = \sup\{f(x) : x_{r-1} \leq x \leq x_r\} = 1$ and $m_r = \inf\{f(x) : x_{r-1} \leq x \leq x_r\} = 0$. Thus $U(f, P) = \sum_{r=1}^{n} M_r\delta_r = 1 \cdot \sum_{r=1}^{n}(x_r - x_{r-1}) = (b - a)$ and $L(f, P) = \sum_{r=1}^{n} m_r\delta_r = 0 \cdot \sum_{r=1}^{n}(x_r - x_{r-1}) = 0$ Since this is true for any partition P, $\int_{\underline{a}}^{b} f(x)\ dx = 0$ and $\int_a^{\bar{b}} f(x)\ dx = 1$. Thus f is not R-integrable.

1.4 A necessary and sufficient condition: Riemann's criteria

THEOREM. 1.22 *A bounded function $f : [a, b] \to \mathbb{R}$ is R-integrable if and only if for every $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that*

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon. \tag{$*$}$$

PROOF. Assume that f is R-integrable. Then $\int_{\underline{a}}^{b} f(x)\ dx = \int_a^{\bar{b}} f(x)\ dx = \int_a^b f(x)\ dx$. Let $\epsilon > 0$ be given. Then since $\int_a^{\bar{b}} f(x)\ dx = \inf\{U(f, P) : P \in \mathcal{P}[a, b]\}$, there exists a partition P_1 of $[a, b]$ such that $U(f, P_1) - \int_a^{\bar{b}} f(x)\ dx < \epsilon/2$, i.e.,

$$U(f, P_1) - \int_a^b f(x)\ dx < \epsilon/2. \tag{I}$$

Similarly, since $\int_{\underline{a}}^{b} f(x)\ dx = \sup\{L(f, P) : P \in \mathcal{P}[a, b]\}$, there exists a partition P_2 of $[a, b]$ such that $\int_{\underline{a}}^{b} f(x)\ dx - L(f, P_2) < \epsilon/2$, i.e.,

$$\int_a^b f(x)\ dx - L(f, P_2) < \epsilon/2. \tag{II}$$

Adding (I) and (II) we have

$$U(f,P_1) - L(f,P_2) < \epsilon. \tag{III}$$

Now let P_ϵ be a partition which is a refinement of both P_1 and P_2. Then $U(f,P_\epsilon) \le U(f,P_1)$ and $L(f,P_2) \le L(f,P_\epsilon)$. Also since $L(f,P_\epsilon) \le U(f,P_\epsilon)$, we have

$$L(f,P_2) \le L(f,P_\epsilon) \le U(f,P_\epsilon) \le U(f,P_1). \tag{IV}$$

From (III) and (IV) we have $U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon$.

Conversely, assume that there exists a partition P_ϵ of $[a,b]$ such that the condition $(*)$ holds. Note that $L(f,P_\epsilon) \le \int_{\underline{a}}^{b} f(x)\ dx \le \int_a^{\bar{b}} f(x)\ dx \le U(f,P_\epsilon)$ which, together with the condition $(*)$, implies that $\int_a^{\bar{b}} f(x)\ dx - \int_{\underline{a}}^{b} f(x)\ dx < \epsilon$, i.e., $\int_a^{\bar{b}} f(x)\ dx < \int_{\underline{a}}^{b} f(x)\ dx + \epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that

$$\int_a^{\bar{b}} f(x)\ dx \le \int_{\underline{a}}^{b} f(x)\ dx.$$

Again it is always true that $\int_{\underline{a}}^{b} f(x)\ dx \le \int_a^{\bar{b}} f(x)\ dx$. Thus, $\int_{\underline{a}}^{b} f(x)\ dx = \int_a^{\bar{b}} f(x)\ dx$, i.e., f is R-integrable. ■

Corollary. 1.23 *If $f : [a,b] \to \mathbb{R}$ is a bounded function and $\{P_n : n \in \mathbb{N}\}$ is a sequence of partitions of $[a,b]$ such that*

$$\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = 0,$$

then f is R-integrable in $[a,b]$ and $\int_a^b f(x)\ dx = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to infty} L(f,P_n)$.

Proof. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $U(f,P_n) - L(f,P_n) < \epsilon$. Thus by Reimann's criteria f is R-integrable in $[a,b]$. Also since for any $n \in \mathbb{N}$, $L(f,P_n) \le \int_{\underline{a}}^{b} f(x)\ dx \le \int_a^{\bar{b}} f(x)\ dx \le U(f,P_n)$, and $\lim_{n\to\infty} U(f,P_n) = \lim_{n\to infty} L(f,P_n)$, it follows that

$$\int_{\underline{a}}^{b} f(x)\ dx = \int_a^{\bar{b}} f(x)\ dx = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n).$$

Thus $\int_a^b f(x)\ dx = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n)$. ■

Example. 1.24 The function $f(x) = x^2,\ 0 \le x \le 1$ is R-integrable in $[0,1]$. Consider the partition $P_n : 0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1$. Since f is increasing in $[0,1]$, we have

$$M_r = \sup\left\{f(x) : \frac{r-1}{n} \le x \le \frac{r}{n}\right\} = \sup\left\{x^2 : \frac{r-1}{n} \le x \le \frac{r}{n}\right\} = \left(\frac{r}{n}\right)^2$$

and

$$m_r = \inf\left\{f(x) : \frac{r-1}{n} \le x \le \frac{r}{n}\right\} = \inf\left\{x^2 : \frac{r-1}{n} \le x \le \frac{r}{n}\right\} = \left(\frac{r-1}{n}\right)^2.$$

Also $\delta_r = \frac{1}{n}$ for $1 \leq r \leq n$. Thus

$$U(f, P_n) = \sum_{r=1}^{n} \left(\frac{r}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

and

$$L(f, P_n) = \sum_{r=1}^{n} \left(\frac{r-1}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{r=1}^{n} (r-1)^2 = \frac{(n-1)(n)(2n-1)}{6n^3} = \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right).$$

Therefore $\lim_{n\to\infty} U(f, P_n) = \frac{1}{3} = \lim_{n\to\infty} L(f, P_n)$. Thus f is R-integrable and

$$\int_a^b f(x)\, dx = \frac{1}{3}.$$

There is another form of Riemann's criterion for which we need the following results.

Lemma. 1.25 *Let f be a bounded function defined on an interval $[a, b]$ such that $|f(x)| \leq k,\ \forall x \in [a, b]$. Let P_1 be a partition of $[a, b]$ and P_2 be a refinement of P_1 which contains at the most p additional points. Then*

$$U(f, P_1) - U(f, P_2) \leq 2kp\delta \ \ \textit{and} \ \ L(f, P_2) - L(f, P_1) \leq 2kp\delta.$$

where $||P_1|| = \delta$.

Proof. Let $P_1 : a = x_0 < x_1 < x_2 < \cdots < x_n = b$. We first consider a partition P_1' which contains a single additional point, i.e., $P_1' : a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i' < x_i < \cdots < x_n = b$. Let

$$M_r = \sup\{f(x) : x_{r-1} \leq x \leq x_r\}, m_r = \inf\{f(x) : x_{r-1} \leq x \leq x_r\}, \quad \text{for } 1 \leq r \leq n$$

and

$$M_i' = \sup\{f(x) : x_{i-1} \leq x \leq x_i'\}, m_i' = \inf\{f(x) : x_{i-1} \leq x \leq x_i'\},$$
$$M_i'' = \sup\{f(x) : x_i' \leq x \leq x_i\}, m_i' = \inf\{f(x) : x_i' \leq x \leq x_i\}.$$

Then it can easily be calculated that

$$\begin{aligned}
U(f, P_1) - U(f, P_1') = \\
& M_i(x_i - x_{i-1}) - \big(M_i'(x_i' - x_{i-1}) - M_i''(x_i - x_i')\big) \\
= \ & \big(M_i(x_i' - x_{i-1}) + M_i(x_i - x_i')\big) - \big(M_i'(x_i' - x_{i-1}) - M_i''(x_i - x_i')\big) \\
= \ & \big(M_i - M_i'\big)(x_i' - x_{i-1}) + \big(M_i - M_i''\big)(x_i - x_i').
\end{aligned}$$

Since $-k \leq f(x) \leq k\ \forall x \in [a, b]$, we have $-k \leq M_i' \leq M_i \leq k$ and $-k \leq M_i'' \leq M_i \leq k$, from which it immediately follows that $M_i - M_i' \leq 2k$ and $M_i - M_i'' \leq 2k$. Thus

$$U(f, P_1) - U(f, P_1') \leq 2k(x_i' - x_{i-1}) + 2k(x_i - x_i') = 2k(x_i - x_{i-1}) \leq 2k\delta.$$

After adding p such points to P_1 we get P_2 and thus obtain $U(f, P_1) - U(f, P_2) \leq 2kp\delta$.

In a similar manner we can show that $L(f, P_2) - L(f, P_1) \leq 2kp\delta$.

THEOREM. 1.26 **Darboux's Theorem:** *If f is bounded function on $[a, b]$, then for each $\epsilon > 0$, there exists a $\delta > 0$ such that for every partition P of $[a, b]$ with $||P|| < \delta$,*

$$U(f, P) - \overline{\int_a^b} f(x)\ dx < \epsilon \quad \textit{and} \quad \underline{\int_a^b} f(x)\ dx - L(f, P) < \epsilon,$$

PROOF. Since f is bounded, there exists $k > 0$ such that $|f(x)| \leq k$ for all $x \in [a.b]$. Let $\epsilon > 0$ be given. Note that $\overline{\int_a^b} f(x)\ dx = \sup\{U(f, P) : P \in \mathcal{P}[a, b]\}$. So there exists a partition $P_1 : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of $[a, b]$ such that

$$U(f, P_1) < \overline{\int_a^b} f(x)\ dx + \epsilon/2. \qquad (*)$$

The partition P_1 has $n+1$ points including a and b. Choose $\delta > 0$ such that $2k(n-1)\delta < \epsilon/2$. Let us consider any partition P of $[a, b]$ with $||P|| < \delta$. Let $P_2 = P \cup P_1$. Then P_2 is a refinement of both P and P_1 and has at the most $n - 1$ additional points than that of P. Thus $U(f, P_2) \leq U(f, P_1)$ and $U(f, P_2) \leq U(f, P)$. Also by the above lemma,

$$\begin{aligned}
& U(f, P) - U(f, P_2) < 2k(n-1)\delta \\
\Rightarrow\ & U(f, P) < 2k(n-1)\delta + U(f, P_2) \\
\Rightarrow\ & U(f, P) < 2k(n-1)\delta + U(f, P_1) < 2k(n-1)\delta + \overline{\textstyle\int_a^b} f(x)\ dx + \epsilon/2 \ \ \text{(using (*) above)} \\
\Rightarrow\ & U(f, P) < \epsilon/2 + \overline{\textstyle\int_a^b} f(x)\ dx + \epsilon/2 \\
\Rightarrow\ & U(f, P) - \overline{\textstyle\int_a^b} f(x)\ dx < \epsilon.
\end{aligned}$$

Similarly we can show the other inequality. ■

THEOREM. 1.27 *A bounded function f defined on $[a, b]$ is R-integrable if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every partition P with $||P|| < \delta$, $U(f, P) - L(f, P) < \epsilon$.*

PROOF. Let us assume that f is R-integrable. Then

$$\overline{\int_a^b} f(x)\ dx = \underline{\int_a^b} f(x)\ dx = \int_a^b f(x)\ dx.$$

Let $\epsilon > 0$ be given. By Daroux's Theorem there exists $\delta > 0$ such that for all partition P with $||P|| < \delta$,

$$U(f, P) - \overline{\int_a^b} f(x)\ dx < \epsilon/2 \text{ and } \underline{\int_a^b} f(x)\ dx - L(, P) < \epsilon/2,$$

which imply that

$$U(f, P) - \int_a^b f(x)\ dx < \epsilon/2 \text{ and } \int_a^b f(x)\ dx - L(, P) < \epsilon/2.$$

Adding we get $U(f, P) - L(f, P) < \epsilon$.

Conversely, Let the condition hold. Since for any partition P of $[a, b]$,

$$L(f,P) \leq \underline{\int_a^b} f(x)\ dx \leq \overline{\int_a^b} f(x)\ dx \leq U(f,P),$$

we have $\overline{\int_a^b} f(x)\ dx - \underline{\int_a^b} f(x)\ dx \leq U(f,P) - L(f,P)$. Since for given $\epsilon > 0$ it is possible to find a partition P with $U(f,P) - L(f,P) < \epsilon$, we have $\overline{\int_a^b} f(x)\ dx - \underline{\int_a^b} f(x)\ dx < \epsilon$, i.e., $\overline{\int_a^b} f(x)\ dx < \underline{\int_a^b} f(x)\ dx + \epsilon$. This is true for every $\epsilon > 0$. Thus $\overline{\int_a^b} f(x)\ dx \leq \underline{\int_a^b} f(x)\ dx$. On the other hand it is always true that $\underline{\int_a^b} f(x)\ dx \leq \overline{\int_a^b} f(x)\ dx$. Thus, $\overline{\int_a^b} f(x)\ dx = \underline{\int_a^b} f(x)\ dx$, i.e., f is R-integrable. ■

1.5 Exercise

1. Let $f(x) = x^2, 0 \leq x \leq 1$. Consider the partitions $P_1 = \{0, .25..5, .75, 1\}$ and $P_2 = \{0, .2, .4, .6, .8, 1\}$. Find $U(f, P_1), L(f, P_1), U(f, P_2)$ and (f, LP_2).
2. Let $f(x) = x(1-x), 0 \leq x \leq 1$. For $n \in \mathbb{N}$, let P_n be the partition $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$. Find $U(f, P_n)$ and $L(f, P_n)$. Hence show that $\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$.

1.6 Some sufficient conditions of integrability

Here we discuss some sufficient conditions of integrability.

Theorem. 1.28 *Every bounded monotone function is R-integrable.*

Proof. Let us consider a monotonic increasing function $f : [a, b] \to \mathbb{R}$. Let $\epsilon > 0$ be any given real number. It is sufficient to find a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. For $n \in \mathbb{N}$ consider the partition $P_n : a = x_0 < x_1 < x_2 < \cdots < x_n = b$, where the points x_i's are placed with equal spacing, i.e., $\delta_i = x_i - x_{i-1} = \frac{b-a}{n}$, $1 \leq i \leq n$. Since the function f is increasing,

$$M_r = \sup\{f(x) : x_{r-1} \leq x \leq x_r\} = f(x_r) \ \text{ and } \ m_r = \inf\{f(x) : x_{r-1} \leq x \leq x_r\} = f(x_{r-1}).$$

Thus

$$\begin{aligned} U(f,P_n) - L(f,P_n) &= \sum_{r=1}^{n} \delta_r (M_r - m_r) \\ &= \sum_{r=1}^{n} \frac{b-a}{n} \left(f(x_r) - f(x_{r-1})\right) \\ &= \frac{b-a}{n} \left(f(b) - f(a)\right). \end{aligned} \tag{1}$$

Now if we choose n such that $n > \frac{(b-a)(f(b)-f(a))}{\epsilon}$, i.e., $\frac{1}{n} < \frac{\epsilon}{(b-a)(f(b)-f(a))}$, then by using relation (1), $U(f, P_n) - L(f, P_n) = (b-a)(f(b) - f(a))\frac{1}{n} < \epsilon$.

Similarly, if f is monotonically decreasing, then $M_r = \sup\{f(x) : x_{r-1} \le x \le x_r\} = f(x_{r-1})$ and $m_r = \inf\{f(x) : x_{r-1} \le x \le x_r\} = f(x_r)$ from which we can get the same conclusion. ■

THEOREM. 1.29 *If $f : [a, b] \to \mathbb{R}$ is continuous, then it is R-integrable on $[a, b]$.*

PROOF. Since f is continuous in a closed interval, it is uniformly continuous. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that for all $x_1, x_2 \in [a, b]$,

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \frac{\epsilon}{(b-a)}. \tag{2}$$

Choose a partition $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ such that $||P|| < \delta$, i.e., for all $r = 1, 2, \cdots, n,\ x_r - x_{r-1} < \delta$. Note that f being continuous on closed interval, it is bounded and attains its bounds. So for any $r,\ 1 \le r \le n$, there exists $x_r', x_r'' \in [x_{r-1}, x_r]$ such that

$$M_r = \sup\{f(x) : x_{r-1} \le x \le x_r\} = f(x_r')$$

and

$$m_r = \inf\{f(x) : x_{r-1} \le x \le x_r\} = f(x_r'').$$

Then $M_r - m_r = f(x_r') - f(x_r'')$. Since $x_r - x_{r-1} < \delta$, we have $x_r' - x_r'' < \delta$ and hence by using the identity (2) we get $|f(x_r') - f(x_r'')| < \epsilon/(b-a)$. Thus

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{r=1}^{n} \delta_r (M_r - m_r) \\
&= \sum_{r=1}^{n} (x_r - x_{r-1})(f(x_r') - f(x_r'')) \\
&< \sum_{r=1}^{n} (x_r - x_{r-1}) \frac{\epsilon}{(b-a)} \\
&= \frac{\epsilon}{(b-a)} (x_n - x_0) \\
&= \frac{\epsilon}{(b-a)} (b-a) = \epsilon
\end{aligned}$$

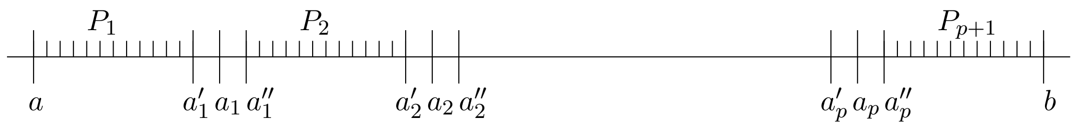
This shows that f is R-integrable. ■

THEOREM. 1.30 *If a bounded function $f : [a, b] \to \mathbb{R}$ has a finite number of points of discontinuity then f is R-integrable on $[a, b]$.*

PROOF. Let $\epsilon > 0$ be given. Let f be discontinuous at the points $a_1, a_2, \cdots, a_p$. We enclose these points with non-intersecting intervals $[a_1', a_1''], [a_2', a_2''], \cdots, [a_p', a_p'']$ such that the total length of these intervals is less than $\epsilon/2(M-m)$ where $M = \sup\{f(x) : a \le x \le b\}$ and $m = \inf\{f(x) : a \le x \le b\}$, i.e.,

$$a_i \in [a_i', a_i''] \text{ for } 1 \le i \le p,\ [a_i', a_i''] \cap [a_j', a_j''] = \emptyset \text{ for } i \ne j$$

$$\text{and } \sum_{i=1}^{p} (a_i'' - a_i') < \frac{\epsilon}{2(M-m)}.$$



Again f, being continuous on each of the $p+1$ intervals $[a, a_1'], [a_1'', a_2'], \cdots, [a_{p-1}'', a_p'], [a_p'', b]$, is integrable there. So there exist partitions $P_1, P_2, \ldots, P_{p+1}$ of $[a, a_1'],\ [a_1'', a_2'], \cdots, [a_p'', b]$ respectively, such that $U(f, P_i) - L(f, P_i) < \frac{\epsilon}{2(p+1)}$ for $1 \le i \le p+1$. Let P be the partition of $[a, b]$ consisting of all the points of P_i, $1 \le i \le p+1$, together with the subintervals $[a, a_1']$, $[a_1'', a_2'], \cdots, [a_p'', b]$.

Then the contribution of $U(f, P) - L(f, P)$ due to a subinterval $[a_i', a_i''], 1 \le i \le p$, is less than or equal to $(M-m)(a_i'' - a_i')$. So the contribution of $U(f, P) - L(f, P)$ due to all the subintervals $[a_i', a_i''], 1 \le i \le p$, is less than or equal to $\sum_{i=1}^{p}(M-m)(a_i'' - a_i') = (M - m)\sum_{i=1}^{p}(a_i'' - a_i') < (M-m)\frac{\epsilon}{2(M-m)} = \frac{\epsilon}{2}$.

Again the contribution of $U(f, P) - L(f, P)$ due to a partition $P_j, 1 \le j \le p+1$, is less than $\frac{\epsilon}{2(p+1)}$. So, the contribution of $U(f, P) - L(f, P)$ due to all the partitions $P_j, 1 \le j \le p+1$, is less than $\sum_{j=1}^{p+1} \frac{\epsilon}{2(p+1)} = (p+1)\frac{\epsilon}{2(p+1)} = \frac{\epsilon}{2}$.

Thus $U(f, P) - L(f, P) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, which shows that f is R- integrable. ■

Theorem. 1.31 *Let f be a bounded function defined on a closed interval $[a, b]$. If the set of points of discontinuity of f has a finite number of limit points then f is R-integrable over $[a, b]$.*

Proof. Let $\epsilon > 0$ be given. Let $M = \sup\{f(x) : a \le x \le b\}$ and $m = \inf\{f(x) : a \le x \le b\}$. Let $a_1, a_2, \ldots, a_p$ be the limit points of the set of points of discontinuity. We enclose these points by non-intersecting intervals $[a_1', a_1''], [a_2', a_2''], \ldots, [a_p', a_p'']$ such that the total length of these intervals is less than $\epsilon/2(M-m)$, i.e.,

$$a_i \in [a_i', a_i''] \text{ for } 1 \le i \le p,\ [a_i', a_i''] \cap [a_j', a_j''] = \emptyset \text{ for } i \ne j \text{ and } \sum_{i=1}^{p}(a_i'' - a_i') < \frac{\epsilon}{2(M-m)}.$$

Let us define $M_i = \sup\{f(x) : a_i' \le x \le a_i''\}$ and $m_i = \inf\{f(x) : a_i' \le x \le a_i''\}$, $1 \le i \le p$. Then $m \le m_i \le M_i \le M$ for $1 \le i \le p$ from which it follows that $M_i - m_i \le M - m, 1 \le i \le p$.

Now each of the $p+1$ intervals $[a', a_1'], [a_1'', a_2'], \ldots, [a_p'', b]$ contains at most finite number of points of discontinuity and hence f is R-integrable on each of these intervals. So there exist partitions $P_1, P_2, \cdots, P_{p+1}$ of $[a, a_1'], [a_1'', a_2'], \ldots, [a_p'', b]$ respectively, such that

$$U(f, P_k) - L(f, P_k) < \frac{\epsilon}{2(p+1)},\ 1 \le k \le p+1.$$

Let P be the partition of $[a, b]$ consisting of all the points of $P_k,\ 1 \le k \le p+1$. Then the contribution of $U(f,P)-L(f,P)$ due to the subintervals $[a,a_1'],[a_1'',a_2'],\ldots,[a_p'',b]$ is

$$\begin{aligned}\sum_{k=1}^{p+1}(U(f,P_k)-L(f,P_k)) &< \sum_{k=1}^{p+1}\frac{\epsilon}{2(p+1)}\\ &= (p+1)\frac{\epsilon}{2(p+1)}\\ &= \frac{\epsilon}{2}. \qquad (3)\end{aligned}$$

On the other hand the contribution of $U(f,P)-L(f,P)$ due to the subintervals $[a_1',a_1'']$, $[a_2',a_2'']$, …, $[a_p',a_p'']$ is

$$\begin{aligned}\sum_{k=1}^{p}(M_k-m_k)(a_k''-a_k') &\le \sum_{k=1}^{p}(M-m)(a_k''-a_k')\\ &= (M-m)\sum_{k=1}^{p}(a_k''-a_k')\\ &< (M-m)\frac{\epsilon}{2(M-m)}\\ &= \frac{\epsilon}{2}. \qquad (4)\end{aligned}$$

Using relations (3) and (4) we conclude that $U(f,P)-L(f,P)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$.

Thus f is R-integrable on $[a,b]$. ■

1.7 Alternative definition of Riemann Integral

Riemann Integration defined in previous sections is due to Daroux. Here we give the original definition provided by Riemann himself and establish the equivalence between the two definitions. We begin with the following definition:

Definition. 1.32 Let f be a function defined on a closed interval $[a,b]$, $P: a=x_0<x_1<x_2<\cdots<x_n=b$ be a partition of $[a,b]$. Then for any choice of points $\xi_i\in[x_{i-1},x_i],\ 1\le i\le n$, the sum $\sum_{i=1}^n f(\xi_i)(x_i-x_{i-1})$ is called a Riemann sum of f.

It immediately follows that for any partition $P: a=x_0<x_1<x_2<\cdots<x_n=b$ and any choice of points $\xi_i\in[x_{i-1},x_i]$, $L(f,P)\le\sum_{i=1}^n f(\xi_i)(x_i-x_{i-1})\le U(f,P)$.

Definition. 1.33 A function f defined on a closed interval $[a,b]$ is called (R-)integrable if there exists a real number I satisfying the following property: For every $\epsilon>0$ there exists a $\delta>0$ such that for every partition $P: a=x_0<x_1<x_2<\cdots<x_n=b$ of $[a,b]$ with $||P||<\delta$ and for every choice of points $\xi_i\in[x_{i-1},x_i], 1\le i\le n$,

$$\left|\sum_{i=1}^n f(\xi_i)(x_i-x_{i-1})-I\right|<\epsilon.$$

The number I is called integral of f over $[a,b]$ and is denoted by $\int_a^b f(x)\,dx$.

1.7.1 Equivalence of the two definitions:

(A) Assume that f is R-integrable in the sense of Darboux.

Then $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$. Let $\epsilon > 0$ be given. Then by Darboux's Theorem there exists $\delta > 0$ such that for all partition $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ with $||P|| < \delta$,

$$\underline{\int_a^b} f(x)\,dx - \epsilon < L(f,P) \text{ and } U(f,P) < \epsilon + \overline{\int_a^b} f(x)\,dx,$$

from which it follows that,

$$\int_a^b f(x)\,dx - \epsilon < L(f,P) \le U(f,P) < \int_a^b f(x)\,dx + \epsilon. \qquad (a)$$

Let us choose for $i = 1, 2, \ldots, n$, $\xi_i \in [x_{i-1}, x_i]$ arbitrarily, then $m_i \le f(\xi_i) \le M_i$ for $1 \le i \le n$, where m_i, M_i have usual meaning. Thus

$$L(f,P) \le \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \le U(f,P).$$

By using (a) we have

$$\int_a^b f(x)\,dx - \epsilon < \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) < \int_a^b f(x)\,dx + \epsilon,$$

i.e.,

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - I \right| < \epsilon.$$

where $I = \int_a^b f(x)\,dx$. Thus f is integrable in the sense of Riemann.

(B) Assume that f is integrable in the sense of Riemann.

First we shall show that f is bounded in $[a,b]$. If possible suppose that f is not bounded. Choose $\epsilon = 1$. Then there exists $\delta > 0$ such that for any partition $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of $[a,b]$ with $||P|| < \delta$ and for any choice of points $\xi_i \in [x_{i-1}, x_i]$, $1 \le i \le n$, $|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - I| < 1$, i.e., $I - 1 < \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) < I + 1$, i.e., $|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})| < |I| + 1$. Now f, being unbounded on $[a,b]$, is unbounded on at least one interval $[x_{k-1}, x_k]$, $1 \le k \le n$. We keep ξ_i fixed for $i \ne k$ and choose ξ_k such that $|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})| > |I| + 1$. This is a contradiction. Thus f is bounded on $[a,b]$.

Now choose any $\epsilon > 0$. Then there exists $\delta > 0$ such that for any partition $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of $[a,b]$ with $||P|| < \delta$ and for any choice of points $\xi_i \in [x_{i-1}, x_i]$, $1 \le i \le n$,

$$I - \frac{\epsilon}{2} < \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) < I + \frac{\epsilon}{2}. \qquad (1)$$

Let $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$ and $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$, $1 \le i \le n$. Then there exists $\alpha_i, \beta_i \in [x_{i-1}, x_i]$, $1 \le i \le n$, such that $f(\alpha_i) - \frac{\epsilon}{2(b-a)} < m_i$ and $M_i < f(\beta_i) + \frac{\epsilon}{2(b-a)}$. Also $m_i \le f(\alpha_i) \le M_i$ and $m_i \le f(\beta_i) \le M_i$. Thus

$$\begin{aligned}\sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1}) &< \sum_{i=1}^{n}\left(m_i + \frac{\epsilon}{2(b-a)}\right)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) + \frac{\epsilon}{2(b-a)}\sum_{i=1}^{n}(x_i - x_{i-1}) \\ &= L(f,P) + \frac{\epsilon}{2}.\end{aligned}$$

Similarly we can show that

$$\sum_{i=1}^{n} f(\beta_i)(x_i - x_{i-1}) > U(f,P) - \frac{\epsilon}{2}.$$

Since the relation (1) is true for any choice of $\xi_i \in [x_{i-1}, x_i]$, in particular taking $\xi_i = \alpha_i$ and $\xi_i = \beta_i$ respectively and using the above relations we get

$$I - \frac{\epsilon}{2} < \sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1}) < L(f,P) + \frac{\epsilon}{2}$$

and

$$U(f,P) - \frac{\epsilon}{2} < \sum_{i=1}^{n} f(\beta_i)(x_i - x_{i-1}) < I + \frac{\epsilon}{2}.$$

i.e.,

$$I - \epsilon < L(f,P) \le U(f,P) < I + \epsilon.$$

Since $L(f,P) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(f,P)$, from the above relation we have

$$I - \epsilon < \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx < I + \epsilon \tag{2}$$

which implies that

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < 2\epsilon.$$

Since this is true for any $\epsilon > 0$, it follows that

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

Thus f is integrable in the sense of Darboux. Let the integral be $\int_a^b f(x)\,dx$. Then

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx. \tag{3}$$

Then using (2) and (3) we have

$$I - \epsilon < \int_a^b f(x)\ dx < I + \epsilon.$$

Since this is true for any $\epsilon > 0$, we have $I = \int_a^b f(x)\ dx$.

This completes the proof. ■

1.8 Properties of Riemann Integral

THEOREM. 1.34 *If f is integrable over $[a, b]$ and k is any constant, then kf is also integrable over $[a, b]$ and $\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$.*

PROOF. If $k = 0$ then kf is constant function and hence integrable.

Assume that $k \neq 0$. Let $\epsilon > 0$ be given. Then, since f is integrable, there exists a partition $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon/|k|$. Let $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$, $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$, $M_i' = \sup\{kf(x) : x_{i-1} \leq x \leq x_i\}$, $m_i' = \inf\{kf(x) : x_{i-1} \leq x \leq x_i\}$, $1 \leq i \leq n$.

Case-I: Assume that $k < 0$. Then $M_i' = km_i$ and $m_i' = kM_i$. Therefore,

$$U(kf, P) = \sum_{i=1}^{n} M_i'(x_i - x_{i-1}) = k\sum_{i=1}^{n} m_i(x_i - x_{i-1}) = kL(f, P)$$

and

$$L(kf, P) = \sum_{i=1}^{n} m_i'(x_i - x_{i-1}) = k\sum_{i=1}^{n} M_i(x_i - x_{i-1}) = kU(f, P).$$

Thus $U(kf, P) - L(kf, P) = k(L(f, P) - U(f, P)) < k(-\frac{\epsilon}{|k|}) = \epsilon$, since $k < 0$. Thus kf is integrable over $[a, b]$.

Case-II: Assume that $k > 0$. Then $M_i' = kM_i$ and $m_i' = km_i$. Therefore,

$$U(kf, P) = \sum_{i=1}^{n} M_i'(x_i - x_{i-1}) = k\sum_{i=1}^{n} M_i(x_i - x_{i-1}) = kU(f, P)$$

and

$$L(kf, P) = \sum_{i=1}^{n} m_i'(x_i - x_{i-1}) = k\sum_{i=1}^{n} m_i(x_i - x_{i-1}) = kL(f, P).$$

Thus $U(kf, P) - L(kf, P) = k(U(f, P) - L(f, P)) < k(\frac{\epsilon}{|k|}) = \epsilon$. Thus kf is integrable over $[a, b]$.

THEOREM. 1.35 *A bounded function f is integrable over $[a, b]$ if and only if for any c, $a < c < b$, f is integrable over $[a, c]$ and $[c, b]$. In the either case $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$.*

PROOF. Assume that f is R-integrable in $[a, b]$. Let $a < c < b$. To show that f is R-integrable in $[a, c]$ and $[c, b]$ and $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$. Let $\epsilon > 0$ be given.

Then the integrability of f over $[a,b]$ implies that there exists a partition P of $[a,b]$ such that $U(f,P)-L(f,P)<\epsilon$. Let $P^*=P\cup\{c\}$. Then P^* is a partition of $[a,b]$ and is a refinement of P. So $L(f,P)\le L(f,P^*)\le U(f,P^*)\le U(f,P)$, i.e., $U(f,P^*)-L(f,P^*)\le U(f,P)-L(f,P)<\epsilon$. Let P_1 and P_2 be the partitions of $[a.c]$ and $[c,b]$ respectively such that $P^*=P_1\cup P_2$. Then $U(f,P^*)=U(f,P_1)+U(f,P_2)$ and $L(f,P^*)=L(f,P_1)+L(f,P_2)$. So,

$$U(f,P_1)-L(f,P_1)+U(f,P_2)-L(f,P_2) \;=\; U(f,P^*)-L(f,P^*)<\epsilon.$$

Thus

$$U(f,P_1)-L(f,P_1)<\epsilon \quad \text{and} \quad U(f,P_2)-L(f,P_2)<\epsilon$$

since $U(f,P_i)-L(f,P_i)>0$ for $i=1,2$.

This shows that f is R-integrable over $[a,c]$ and $[c,b]$. Also since $U(f,P^*)=U(f,P_1)+U(f,P_2)$, taking infimum on both sides we get $\int_a^{\bar{b}} f(x)\,\mathrm{d}x=\int_a^{\bar{c}} f(x)\,\mathrm{d}x+\int_c^{\bar{b}} f(x)\,\mathrm{d}x$, i.e., $\int_a^b f(x)\,\mathrm{d}x=\int_a^c f(x)\,\mathrm{d}x+\int_c^b f(x)\,\mathrm{d}x$.

Conversely assume that f is R-integrable over $[a,c]$ and $[c,b]$. Let $\epsilon.0$ be arbitrary. Then there exist partitions P_1 and P_2 of $[a,c]$ and $[c,b]$ respectively such that $U(f,P_i)-L(f,P_i)<\epsilon/2$ for $i=1,2$. Let $P=P_1\cup P_2$. Then P is a partition of $[a,b]$. Also $U(f,P)=U(f,P_1)+U(f,P_2)$ and $L(f,P)=L(f,P_1)+L(f,P_2)$. Thus

$$U(f,P)-L(f,P) \;=\; U(f,P_1)-L(f,P_1)+U(f,P_2)-L(f,P_2)<\epsilon/2+\epsilon/2=\epsilon.$$

This shows that f is R-integrable over $[a,b]$. Also by taking infimum on both sides of $U(f,P)=U(f,P_1)+U(f,P_2)$ we get $\int_a^{\bar{b}} f(x)\,\mathrm{d}x=\int_a^{\bar{c}} f(x)\,\mathrm{d}x+\int_c^{\bar{b}} f(x)\,\mathrm{d}x$, i.e., $\int_a^b f(x)\,\mathrm{d}x=\int_a^c f(x)\,\mathrm{d}x+\int_c^b f(x)\,\mathrm{d}x$.

This completes the proof. ■

THEOREM. 1.36 *If f and g are integrable over $[a,b]$ then $f\pm g$ are also integrable over $[a,b]$ and $\int_a^b (f\pm g)(x)\ dx=\int_a^b f(x)\ dx\pm\int_a^b g(x)\ dx$.*

PROOF. Assume that f and g are R-integrable over $[a,b]$. Let $\epsilon>0$ be any given real number. Then there exist partitions P_1 and P_2 of $[a,b]$ such that

$$U(f,P_1)-L(f,P_1)<\epsilon/2 \quad \text{and} \quad U(g,P_2)-L(g,P_2)<\epsilon/2.$$

Let $P=P_1\cup P_2$. Then P is a refinement of both P_1 and P_2. So,

$$\begin{aligned} U(f,P)-L(f,P)<U(f,P_1)-L(f,P_1) &< \epsilon/2 \\ U(g,P)-L(g,P)<U(g,P_2)-L(g,P_2) &< \epsilon/2. \end{aligned}$$

Let $P := a = x_0 < x_1 < x_2 < \cdots < x_n = b$. For $1 \le r \le n$ we define

$$\begin{aligned} M'_r &= \sup\{f(x) : x_{r-1} \le x \le x_4\} \text{ and } m'_r = \inf\{f(x) : x_{r-1} \le x \le x_4\} \\ M''_r &= \sup\{g(x) : x_{r-1} \le x \le x_4\} \text{ and } m''_r = \inf\{g(x) : x_{r-1} \le x \le x_4\} \\ M_r &= \sup\{(f+g)(x) : x_{r-1} \le x \le x_4\} \text{ and } m_r = \inf\{(f+g)(x) : x_{r-1} \le x \le x_4\}. \end{aligned}$$

Then obviously $M_r \le M'_r + M''_r$ and $m_r \ge m'_r + m''_r$ for $1 \le r \le n$. Thus,

$$\begin{aligned} \int_a^{\bar{b}} (f+g)(x)\,\mathrm{d}x &\le U(f+g,P) = \sum_{r=1}^{n} M_r(x_r - x_{r-1}) \\ &\le \sum_{r=1}^{n} (M'_r + M''_r)(x_r - x_{r-1}) \\ &= \sum_{r=1}^{n} M'_r(x_r - x_{r-1}) + \sum_{r=1}^{n} M''_r(x_r - x_{r-1}) \\ &= U(f,P) + U(g,P) \end{aligned}$$

and

$$\begin{aligned} \int_{\underline{a}}^{b} (f+g)(x)\,\mathrm{d}x &\ge L(f+g,P) = \sum_{r=1}^{n} m_r(x_r - x_{r-1}) \\ &\ge \sum_{r=1}^{n} (m'_r + m''_r)(x_r - x_{r-1}) \\ &= \sum_{r=1}^{n} m'_r(x_r - x_{r-1}) + \sum_{r=1}^{n} m''_r(x_r - x_{r-1}) \\ &= L(f,P) + L(g,P). \end{aligned}$$

Thus

$$\begin{aligned} \int_a^{\bar{b}} (f+g)(x)\,\mathrm{d}x - \int_{\underline{a}}^{b} (f+g)(x)\,\mathrm{d}x &\le (U(f,P) + U(g,P)) - (L(f,P) + L(g,P)) \\ &= (U(f,P) - L(f,P)) + (U(g,P)) - L(g,P)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

i.e., $\int_a^{\bar{b}} (f+g)(x)\,\mathrm{d}x < \int_{\underline{a}}^{b} (f+g)(x)\,\mathrm{d}x + \epsilon$. Since this is true for every $\epsilon > 0$ we get

$$\int_a^{\bar{b}} (f+g)(x)\,\mathrm{d}x \le \int_{\underline{a}}^{b} (f+g)(x)\,\mathrm{d}x.$$

On the other hand it is always true that

$$\int_{\underline{a}}^{b} (f+g)(x)\,\mathrm{d}x \le \int_a^{\bar{b}} (f+g)(x)\,\mathrm{d}x.$$

Combining the above two relations,

$$\int_a^{\bar{b}} (f+g)(x)\,\mathrm{d}x = \int_a^{\bar{b}} (f+g)(x)\,\mathrm{d}x.$$

i.e., $f+g$ is R-integrable over $[a,b]$.

Also taking limit as $||P|| \to 0$ on both sides of the relations,

$$\begin{aligned} U(f+g,P) &\le U(f,P)+U(g,P) \\ L(f+g,P) &\ge L(f,P)+L(g,P) \end{aligned}$$

we get

$$\begin{aligned} \int_a^{\bar{b}} (f+g)(x)\,\mathrm{d}x &\le \int_a^{\bar{b}} f(x)\,\mathrm{d}x + \int_a^{\bar{b}} g(x)\,\mathrm{d}x \\ \int_a^{\bar{b}} (f+g)(x)\,\mathrm{d}x &\ge \int_a^{\bar{b}} f(x)\,\mathrm{d}x + \int_a^{\bar{b}} g(x)\,\mathrm{d}x \end{aligned}$$

i.e.,

$$\begin{aligned} \int_a^{b} (f+g)(x)\,\mathrm{d}x &\le \int_a^{b} f(x)\,\mathrm{d}x + \int_a^{b} g(x)\,\mathrm{d}x \\ \int_a^{b} (f+g)(x)\,\mathrm{d}x &\ge \int_a^{b} f(x)\,\mathrm{d}x + \int_a^{b} g(x)\,\mathrm{d}x. \end{aligned}$$

Hence,

$$\int_a^{b} (f+g)(x)\,\mathrm{d}x = \int_a^{b} f(x)\,\mathrm{d}x + \int_a^{b} g(x)\,\mathrm{d}x$$

This completes the proof. ■

THEOREM. 1.37 *If f and g are integrable over $[a,b]$ then fg is also integrable over $[a,b]$.*

PROOF. Since f and g are R-integrable, they are bounded. So there exists $M>0$ such that $\forall\ x \in [a,b], |f(x)| < M, |g(x)| < M$. Then for any $x \in [a,b]$, $|(fg)(x)| = |f(x)\ g(x)| = |f(x)|\ |g(x)| < M^2$. Thus fg is also bounded.

Let $\epsilon > 0$ be any real. The integrability of f and g implies that there exist partition P_1 and P_2 of $[a,b]$ such that

$$U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2M} \quad \text{and} \quad U(g,P_2) - L(g,P_2) < \frac{\epsilon}{2M}.$$

Let P be a partition of $[a,b]$ which is a refinement of both P_1 and P_2. Then

$$\begin{aligned} U(f,P) - L(f,P) &\le U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2M} \\ U(g,P) - L(g,P) &\le U(g,P_2) - L(g,P_2) < \frac{\epsilon}{2M}. \end{aligned}$$

Let $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$. For $1 \le r \le n$ we define,

$$\begin{aligned} M_r &= \sup\{(fg)(x) : x \in [x_{r-1}, x_r]\} & m_r &= \inf\{(fg)(x) : x \in [x_{r-1}, x_r]\} \\ M_r' &= \sup\{f(x) : x \in [x_{r-1}, x_r]\} & m_r' &= \inf\{f(x) : x \in [x_{r-1}, x_r]\} \\ M_r'' &= \sup\{g(x) : x \in [x_{r-1}, x_r]\} & m_r'' &= \inf\{g(x) : x \in [x_{r-1}, x_r]\}. \end{aligned}$$

Now for any $x, y \in [x_{r-1}, x_r], 1 \le r \le n$, we have,

$$\begin{aligned} |(fg)(x) - (fg)(y)| &= |f(x)g(x) - f(y)g(y)| \\ &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &\le |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)| \\ &\le M(M_r'' - m_r'') + M(M_r' - m_r'). \end{aligned}$$

Thus $M_r - m_r \le M(M_r'' - m_r'') + M(M_r' - m_r')$. Multiplying both sides by $\delta_r = (x_r - x_{r-1})$ and taking summation from $r = 1$ to n we get,

$$\begin{aligned} \sum_{r=1}^{n} (M_r - m_r)\delta_r &\le M\sum_{r=1}^{n}(M_r'' - m_r'')\delta_r + M\sum_{r=1}^{n}(M_r' - m_r')\delta_r \\ \text{i.e., } U(fg, P) - L(fg, P) &\le M\{U(g, P) - L(g, P)\} + M\{U(f, P) - L(f, P)\} \\ &< M.\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon. \end{aligned}$$

Thus fg is R-integrable. ■

THEOREM. 1.38 *If f and g are R-integrable over $[a, b]$ and there exists $k > 0$ such the $|g(x)| \ge k \ \forall x \in [a, b]$, then $\frac{f}{g}$ is R-integrable over $[a, b]$.*

PROOF. Since f is integrable over $[a, b]$ it is bounded there, so there exists $M > 0$ such that $|f(x)| \le M \ \forall x$ in $[a, b]$. Also $|g(x)| \ge k \ \forall x \in [a, b]$. Thus $\left|\frac{f}{g}(x)\right| = \frac{|f(x)|}{|g(x)|} \le \frac{M}{k}$. Thus $\frac{f}{g}$ is bounded.

Since f, g are R-integrable there exists partitions P_1, P_2 of $[a, b]$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2M}k^2 \quad \text{and} \quad U(g, P_2) - L(g, P_2) < \frac{\epsilon}{2}k.$$

Let P be a partition of $[a, b]$ which is a refinement of both P_1 and P_2. Then

$$U(f, P) - L(f, P) < \frac{\epsilon}{2M}k^2 \quad \text{and} \quad U(g, P) - L(g, P) < \frac{\epsilon}{2}k.$$

Let $P : a = x_0 < x_1 < x_2 \cdots < x_n = b$. We define

$$\begin{aligned} M_r &= \sup\{(f/g)(x) : x \in [x_{r-1}, x_r]\} & m_r &= \inf\{(f/g)(x) : x \in [x_{r-1}, x_r]\} \\ M_r' &= \sup\{f(x) : x \in [x_{r-1}, x_r]\} & m_r' &= \inf\{f(x) : x \in [x_{r-1}, x_r]\} \\ M_r'' &= \sup\{g(x) : x \in [x_{r-1}, x_r]\} & m_r'' &= \inf\{g(x) : x \in [x_{r-1}, x_r]\}. \end{aligned}$$

Now for $1 \le r \le n$, for $x, y \in [x_{r-1}, x_r]$,

$$\begin{aligned}
\left|\frac{f}{g}(x) - \frac{f}{g}(y)\right| &= \left|\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right| = \left|\frac{f(x)g(y) - g(x)f(y)}{g(x)g(y)}\right| \\
&= \left|\frac{f(x)g(y) - f(x)g(x) + f(x)g(x) - g(x)f(y)}{g(x)g(y)}\right| \\
&\le \frac{|f(x)||g(y) - g(x)| + |g(x)||f(x) - f(y)|}{|g(x)||g(y)|} \\
&= \left|\frac{f(x)}{g(x)}\frac{1}{g(y)}\right| |g(y) - g(x)| + \left|\frac{1}{g(y)}\right| |f(x) - f(y)| \\
&< \frac{M}{k}\frac{1}{k}(M''_r - m''_r) + \frac{1}{k}(M'_r - m'_r).
\end{aligned}$$

Since this is true for all x, y in $[x_{r-1}, x_r]$,

$$M_r - m_r \le \frac{M}{k}\frac{1}{k}(M''_r - m''_r) + \frac{1}{k}(M'_r - m'_r).$$

Multiplying both sides by $\Delta_r = (x_r - x_{r-1})$ and taking summation from $r = 1$ to n we get,

$$\begin{aligned}
\sum_{r=1}^{n}(M_r - m_r)\Delta_r &\le \frac{M}{k^2}\sum_{r=1}^{n}(M''_r - m''_r)\Delta_r + \frac{1}{k}\sum_{r=1}^{n}(M'_r - m'_r)\Delta_r \\
U(\frac{f}{g}, P) - L(\frac{f}{g}, P) &\le \frac{M}{k^2}(U(g, P) - L(g, P)) + \frac{1}{k}(U(f, P) - L(f, P)) \\
&< \frac{M}{k^2}\frac{\epsilon}{2M}k^2 + \frac{1}{k}\frac{\epsilon}{2}k \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus $\frac{f}{g}$ is R-integrable over $[a, b]$. ■

THEOREM. 1.39 *If f is R-integrable over $[a, b]$ then $|f|$ is also R-integrable over $[a, b]$ and*

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x.$$

PROOF. Since f is bounded, there exists $k > 0$ such that $-k \le f(x) \le k,\ \forall x \in [a, b]$. Thus $|f(x)| \le k\ \forall x \in [a, b]$, i.e., $|f|(x) \le k,\ \forall x \in [a, b]$. Thus $|f|$ is bounded.

Let $\epsilon > 0$ be given. Since f is integrable over $[a, b]$ there exists a partition $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Let

$$\begin{aligned}
M_r &= \sup\{f(x) : x \in [x_{r-1}, x_r]\} \qquad m_r = \inf\{f(x) : x \in [x_{r-1}, x_r]\} \\
M'_r &= \sup\{|f|(x) : x \in [x_{r-1}, x_r]\} \qquad m'_r = \inf\{|f|(x) : x \in [x_{r-1}, x_r]\}.
\end{aligned}$$

Now for $1 \le r \le n$, for any $x, y \in [x_{r-1}, x_r]$,

$$|\,|f|(x) - |f|(y)\,| = |\,|f(x)| - |f(y)|\,| \le |f(x) - f(y)|$$

which implies that $M_r' - m_r' \leq M_r - m_r,\ \ 1 \leq r \leq n$.

Thus

$$\begin{aligned} U(|f|, P) - L(|f|, P) &= \sum_{r=1}^{n}(M_r' - m_r')(x_r - x_{r-1}) \leq \sum_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1}) \\ &= U(f, P) - L(f, P) < \epsilon. \end{aligned}$$

This shows that $|f|$ is R-integrable.

Also for any x in $[a, b]$,

$$\begin{aligned} -|f(x)| \leq f(x) \leq |f(x)| \ &\Rightarrow \int_a^b -|f(x)|\,\mathrm{d}x \leq \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x \\ \Rightarrow \ -\int_a^b |f(x)|\,\mathrm{d}x &\leq \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x. \end{aligned}$$

Thus $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f|(x)\,\mathrm{d}x$. ■

The converse of the above theorem is not true.

EXAMPLE. 1.40 Let us define $f : [0, 1] \to \mathbb{R}$ by $f(x) = 1$ when x is rational and $f(x) = -1$ when x is irrational. Then obviously f is not integrable on $[0, 1]$, however $|f|(x) = 1$ for all x in $[0, 1]$ which is constant and hence integral.

1.9 Fundamental Theorem of Integral Calculus

In practice, when we integrate a function f over an interval $[a, b]$, we first find an anti-derivative F of the function f such that $F' = f$ on $[a, b]$. Then we take $F(b) - F(a)$ as the value of the integral $\int_a^b f(x)\,\mathrm{d}x$. The theory behind this practice is the fundamental theorem if integral calculus.

THEOREM. 1.41 (FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS) *Let $f : [a, b] \to \mathbb{R}$ be an integrable function and $F : [a, b] \to \mathbb{R}$ be a function having the following properties:*

1. *F is continuous on $[a, b]$*
2. *F is differentiable on (a, b) and $F'(x) = f(x)\,\forall x \in (a, b)$.*

Then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$.

PROOF. Let $\epsilon > 0$ be arbitrary. Since f is integrable, there exists a partition $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ such that for any choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left|\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) - \int_a^b f(x)\,\mathrm{d}x\right| \ < \ \epsilon. \tag{5}$$

Also we can write by using Lagrange's Mean Value Theorem,

$$\begin{aligned} F(b) - F(a) &= \sum_{r=1}^{n} (F(x_r) - F(x_{r-1})) && (6) \\ &= \sum_{r=1}^{n} (x_r - x_{r-1}) F'(\xi_r), \text{for some } \xi_r \in (x_{r-1}, x_r), 1 \le r \le n \\ &= \sum_{r=1}^{n} (x_r - x_{r-1}) f(\xi_r). \text{ since } F' = f \text{ on } (a,b). && (7) \end{aligned}$$

Thus by 5 and 7, it follows that $\left| F(b) - F(a) - \int_a^b f(x)\,\mathrm{d}x \right| < \epsilon$. Since $\epsilon > 0$ is arbitrary, we have

$$F(b) - F(a) = \int_a^b f(x)\,\mathrm{d}x.$$

This completes the proof. ■

1.10 Worked out problems

1. If $f : [a,b] \to \mathbb{R}$ be a bounded function prove that f is Riemann integrable over $[a,b]$ if and only if for any $\epsilon > 0$ there is a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.

2. Give an example with proper justification of a Riemann integrable function which has no primitive.

3. Prove that $f : [0,3] \to \mathbb{R}$ defined by $f(x) = x + [x]$ is integrable.

4. Give an example, with proper justifications, of a discontinuous function which has a primitive.

5. If a function $f : [a,b] \to \mathbb{R}$ be integrable and $f(x) \ge 0$ for $x \in [a,b]$ and there exists a point $c \in [a,b]$, such that f is continuous at c with $f(c) > 0$, then prove that $\int_a^b f > 0$.

6. Let f be continuous on $[a,b]$ and for each $\alpha, \beta, a \le \alpha < \beta < b$,

$$\int_\alpha^\beta f(x)\ dx = 0.$$

   Prove that f is identically zero on $[a,b]$.

7. If a function $f : [a,b] \to \mathbb{R}$ be bounded and for every $c \in (a,b)$, f is integrable on $[c,b]$, then prove that f is integrable on $[a,b]$.

8. Give an example of a function $f : [0,1] \to \mathbb{R}$ which is integrable on $[c,1]$, $0 < c < 1$ but not integrable on $[0,1]$.

9. Find the lower and upper integrals of the function.

$$\begin{aligned} f(x) &= 1,\ x \in \mathbb{Q} \cap [0,1] \\ &= 0,\ x \in (\mathbb{R} - \mathbb{Q}) \cap [0,1] \end{aligned}$$

10. For bounded function f defined on an interval $[a, b]$ and any two partitions P_1, P_2 of $[a, b]$ show that $L(f, P_1) \leq U(f, P_2)$.

11. Prove that a continuous function f defined on a closed interval $[a, b]$ is integrable in the sense of Riemann.

12. A function $f : [0, 1] \to \mathbb{R}$ is defined by

$$\begin{aligned} f(x) &= \frac{1}{3^n},\ \frac{1}{3^{n+1}} < x \leq \frac{1}{3^n}, n = 0, 1, 2, \ldots \\ &= 0,\ x = 0. \end{aligned}$$

Show that f is integrable in the sense of Riemann and $\int_0^1 f(x)\ dx = \frac{3}{4}$

13. Using Mean Value Theorem of Integral Calculus prove that

$$\frac{\pi^3}{24} \leq \int_0^{\pi} \frac{x^3}{5 + 3\cos x} dx \leq \frac{\pi^3}{6}.$$

2 Improper Integral

So far we have studied the theory of integration, we have assumed that the following conditions are satisfied:

1. the integrand function is bounded over the interval of integration,
2. the interval of integration is bounded.

When any one or both of the above conditions are not satisfied, we still try to integrate the function by using the concept of limit. An integral of this type, when exists, is known as *improper integral*. There are two types of improper integrals:

1. When the range of integration is finite but the integrand has an infinite discontinuity at any of the end points or in any interior point. This can be of the form:
   (a) $\int_a^b f(x)\,\mathrm{d}x$ where f has an infinite discontinuity at $x = a$, for example, $\int_0^1 \frac{dx}{x^2}$.
   (b) $\int_a^b f(x)\,\mathrm{d}x$ where f has an infinite discontinuity at $x = b$, for example, $\int_1^2 \frac{dx}{(x-2)^3}$.
   (c) $\int_a^b f(x)\,\mathrm{d}x$ where f has an infinite discontinuity at $x = c$ where c is a point lying between a and b. For example, $\int_0^2 \frac{dx}{(x-1)^4}$.
   (d) $\int_a^b f(x)\,\mathrm{d}x$ where f has a finite number of infinite discontinuities, say, at $x = c_1, c_2, \ldots, c_k$ where $a \le c_1 < c_2 < \cdots < c_k \le b$. For example, $\int_0^{2\pi} \frac{\sin 2x}{\sin 2x - \cos 2x} dx$. Here the integrand has infinite discontinuities at $\pi/8, 5\pi/8, 9\pi/8$ and at $13\pi/8$.
2. When the range of integration is infinite, the integrand being a bounded function.

The combination of the above two types is also possible, i.e., when the range of integration is infinite and also the integrand is an unbounded function.

2.1 When range of integration is finite:

We consider the improper integral $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ when $f(x)$ has a point of infinite discontinuity only at $x = a$. We take the integral $\displaystyle\int_{a+\epsilon}^b f(x)\,\mathrm{d}x$ where $0 < \epsilon < b - a$. This is a proper integral (as there is no other point of infinite discontinuity of f in the range). Suppose that $\displaystyle\int_{a+\epsilon}^b f(x)\,\mathrm{d}x$ exists and equal to $\phi(\epsilon)$. If $\lim_{\epsilon\to 0+0} \phi(\epsilon)$ exists and is finite, say equal to l, we say the improper integral $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ exists or converges at $x = a$ and write as

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\epsilon\to 0+} \int_{a+\epsilon}^b f(x)\,\mathrm{d}x = l.$$

Again, if $x = b$ be the only point of infinite discontinuity of f in the finite integral $[a, b]$, then $\int_a^b f(x)\,\mathrm{d}x$ exists or converges at $x = b$ if $\lim_{\epsilon\to 0+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x$ exists, $0 < \epsilon < b - a$. We then write,

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\epsilon\to 0+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x.$$

Example. 2.1 $\int_0^1 \frac{dx}{x^2}$. Here $f(x) = \frac{1}{x^2}$ has only one point of infinite discontinuity at $x = 0$. Then,

$$\int_0^1 \frac{dx}{x^2} = \lim_{\epsilon\to 0+} \int_\epsilon^1 \frac{dx}{x^2} = \lim_{\epsilon\to 0+} \left[-\frac{1}{x} \right]_\epsilon^1 = \lim_{\epsilon\to 0+} \left\{ \frac{1}{\epsilon} - 1 \right\} = \infty.$$

Thus the integral $\int_0^1 \frac{dx}{x^2}$ does not converge.

Example. 2.2 $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. Here $f(x) = \frac{1}{\sqrt{1-x^2}}$. $x = 1$ is a point of infinite discontinuity of f. Then

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1-x^2}} &= \lim_{\epsilon\to 0+} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}} = \lim_{\epsilon\to 0+} \left[\sin^{-1} x \right]_0^{1-\epsilon} \\ &= \lim_{\epsilon\to 0+} \left\{ \sin^{-1}(1-\epsilon) - \sin^{-1} 0 \right\} = \sin^{-1} 1 = \frac{\pi}{2} \end{aligned}$$

When both of a and b are the only points of infinite discontinuity of f in the finite range $[a, b]$, we take any point c where $a < c < b$ and consider the integrals $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$. The integral $\int_a^b f(x)\,\mathrm{d}x$ converges if $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$ converge at $x = a$ and $x = b$ respectively.

It is to be noted that the result is independent of the choice of the point $x = c$. (Proof is beyond the scope of this note).

Example. 2.3 $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$. Here $f(x) = \frac{1}{\sqrt{x(1-x)}}$, both of $x = 0$ and $x = 1$ are the points of infinite discontinuity of f. We take $c = \frac{1}{2}$ and consider the integrals $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}}$ and $\int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{x(1-x)}}$. Now,

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} = \lim_{\epsilon\to 0+} \int_\epsilon^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} = \lim_{\epsilon\to 0+} \int_\epsilon^{\frac{1}{2}} \frac{dx}{\sqrt{x - x^2)}}$$

$$= \lim_{\epsilon\to 0+}\int_{\epsilon}^{\frac{1}{2}} \frac{dx}{\sqrt{(\frac{1}{2})^2-(x-\frac{1}{2})^2}} = \lim_{\epsilon\to 0+}\left[\ \sin^{-1}(2x-1)\ \right]_{\epsilon}^{\frac{1}{2}}$$

$$= \lim_{\epsilon\to 0+}\left\{\sin^{-1}0-\sin^{-1}(2\epsilon-1)\right\} = -\sin^{-1}(-1) = \sin^{-1}1 = \frac{\pi}{2}.$$

Thus the integral $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}}$ converges to $\frac{\pi}{2}$.

Also,

$$\int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{x(1-x)}} = \lim_{\epsilon\to 0+}\int_{\frac{1}{2}}^{1-\epsilon} \frac{dx}{\sqrt{x(1-x)}} = \lim_{\epsilon\to 0+}\int_{\frac{1}{2}}^{1-\epsilon} \frac{dx}{\sqrt{x-x^2)}}$$

$$= \lim_{\epsilon\to 0+}\int_{\frac{1}{2}}^{1-\epsilon} \frac{dx}{\sqrt{(\frac{1}{2})^2-(x-\frac{1}{2})^2}} = \lim_{\epsilon\to 0+}\left[\ \sin^{-1}(2x-1)\ \right]_{\frac{1}{2}}^{1-\epsilon}$$

$$= \lim_{\epsilon\to 0+}\left\{\sin^{-1}(1-2\epsilon)-\sin^{-1}0\right\} = \sin^{-1}1 = \frac{\pi}{2}.$$

Thus the integral $\displaystyle\int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{x(1-x)}}$ converges to $\frac{\pi}{2}$. Hence, $\displaystyle\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ converges and

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} + \int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

2.2 When range of integration is infinite:

Consider the integral $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$. Let f be bounded and integrable over $[a, X]$ for every $X \geq a$. Then $\displaystyle\int_a^X f(x)\,\mathrm{d}x$ exists and equal to, say, $\phi(X)$. If the limit $\displaystyle\lim_{X\to\infty}\phi(X)$ exists and finite, say l, then we say that the improper integral $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ converges with value l. Therefore,

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{X\to\infty}\int_a^X f(x)\,\mathrm{d}x$$

Consider the integral $\displaystyle\int_{-\infty}^b f(x)\,\mathrm{d}x$. Let f be bounded and integrable over $[X, b]$ where $X \leq b$. If the limit $\displaystyle\lim_{X\to-\infty}\int_X^b f(x)\,\mathrm{d}x$ exists and has finite value we say that the integral $\displaystyle\int_{-\infty}^b f(x)\,\mathrm{d}x$ converges. We write,

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{X\to\infty}\int_X^b f(x)\,\mathrm{d}x$$

Consider the integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$. Take any number c and consider the integrals $\int_{-\infty}^{c} f(x)\,\mathrm{d}x$ and $\int_{c}^{\infty} f(x)\,\mathrm{d}x$. If both the integrals Converge, then we say that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ converges and write,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x &= \int_{-\infty}^{c} f(x)\,\mathrm{d}x + \int_{c}^{\infty} f(x)\,\mathrm{d}x \\ &= \lim_{X\to-\infty}\int_{X}^{c} f(x)\,\mathrm{d}x + \lim_{X'\to\infty}\int_{c}^{X'} f(x)\,\mathrm{d}x.\end{aligned}$$

It is to be noted that the result is independent of the choice of c. (Proof is beyond the scope of this note).

Example. 2.4 Evaluate $\int_0^{\infty} \frac{dx}{1+x^2}$.

$$\begin{aligned}\int_0^{\infty}\frac{dx}{1+x^2} &= \lim_{X\to\infty}\int_0^{X}\frac{dx}{1+x^2} = \lim_{X\to\infty}\left[\tan^{-1}x\right]_0^X \\ &= \lim_{X\to\infty}\left[\tan^{-1}X-\tan^{-1}0\right] = \lim_{X\to\infty}\tan^{-1}X = \frac{\pi}{4}\end{aligned}$$

2.2.1 Problems

(i) $\int_0^1 \frac{dx}{1-x}$ (ii) $\int_0^{\infty}\frac{x}{x^2+4}\,\mathrm{d}x$ (iii) $\int_0^{\infty}\frac{dx}{x^2-1}$

(vi) $\int_0^{\infty}\frac{x\,\mathrm{d}x}{(1+x^2)^2}$ (v) $\int_0^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x, a,b>0$ (vi) $\int_{-\infty}^{\infty}\frac{x}{x^4+1}\,\mathrm{d}x$

(vii) $\int_0^{\infty}\frac{dx}{x^2+2x+2}$ (viii) $\int_0^{\infty}\frac{x^2\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)}, a,b>0$

Answer: (i) does not converge, (ii) does not converge, (iii) $\frac{1}{2}\log 2$, (iv) $\frac{1}{4}$, (v) 0, (vi) $\frac{\pi}{2ab(a+b)}$, (vii) $\frac{\pi}{2(a+b)}$, (viii) Does not converge.

2.3 Tests for Convergence of Improper Integral

We begin with a few useful examples.

Example. 2.5 1. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if $n<1$ and is divergent if $n\geq 1$.

The integral is proper if $n\leq 0$. for $n>0$ and $n\neq 1$,

$$\begin{aligned}\int_a^b\frac{dx}{(x-a)^n} &= \lim_{\epsilon\to 0+}\int_{a+\epsilon}^b\frac{dx}{(x-a)^n} = \lim_{\epsilon\to 0+}\left[\frac{1}{-n+1}(x-a)^{-n+1}\right]_{a+\epsilon}^b \\ &= \frac{1}{1-n}\cdot\lim_{\epsilon\to 0+}\left[(b-a)^{1-n}-\epsilon^{1-n}\right].\end{aligned}$$

When $n > 1$ then $\lim_{\epsilon \to 0+} \epsilon^{1-n} = \infty$ and when $n < 1$ then $\lim_{\epsilon \to 0+} \epsilon^{1-n} = 0$. Thus,

$$\begin{aligned} \int_a^b \frac{dx}{(x-a)^n} &= \frac{1}{1-n} \cdot (b-a)^{1-n} \quad \text{if } n < 1 \\ &= \infty \text{ if } n > 1. \end{aligned}$$

For $n = 1$,

$$\begin{aligned} \int_a^b \frac{dx}{x-a} &= \lim_{\epsilon \to 0+} \int_{a+\epsilon}^b \frac{dx}{x-a} = \lim_{\epsilon \to 0+} [\log(x-a)]_{a+\epsilon}^b \\ &= \lim_{\epsilon \to 0+} [\log(b-a) - \log \epsilon] = \infty. \end{aligned}$$

Hence the integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if $n < 1$ and divergent if $n \geq 1$.

2. The integral $\int_a^\infty \frac{dx}{x^p}, a > 0$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

   For $p \neq 1$ the integral is evaluated as,

$$\begin{aligned} \int_a^\infty \frac{dx}{x^p} &= \lim_{X \to \infty} \int_a^X \frac{dx}{x^p} = \frac{1}{1-p} \cdot \lim_{X \to \infty} \left[x^{1-p} \right]_a^X \\ &= \frac{1}{1-p} \cdot \lim_{X \to \infty} [X^{1-p} - a^{1-p}] = \infty \quad \text{if } p < 1 \\ &= \frac{1}{p-1} a^{1-p} \quad \text{if } p > 1. \end{aligned}$$

   For $p = 1$ the integral becomes,

$$\int_a^\infty \frac{dx}{x} = \lim_{X \to \infty} \int_a^X \frac{dx}{x} = \lim_{X \to \infty} [\log x]_a^X = \lim_{X \to \infty} [\log X - \log a] = \infty.$$

   Hence the integral is convergent if $p > 1$ and is divergent if $p \leq 1$.

Problem. 2.6 Test the convergence of the integral $\int_a^b \frac{dx}{(b-x)^n}$.

Theorem. 2.7 (Comparison test:) *If f and g are two non-negative functions defined on $(a, b]$, having the only infinite discontinuity at a and $f \leq g$ on $(a, c]$ for some $c, a < c \leq b$, then (i) if $\int_a^b g(x)\,\mathrm{d}x$ is convergent then so is $\int_a^b f(x)\,\mathrm{d}x$, (ii) if $\int_a^b f(x)\,\mathrm{d}x$ is divergent then so is $\int_a^b g(x)\,\mathrm{d}x$.*

Proof. Since $\lim_{\epsilon \to 0} \int_{a+\epsilon}^b f(x)\,\mathrm{d}x = \lim_{\epsilon \to 0} \int_{a+\epsilon}^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$, and the last integral is proper one, without any loss of generality we may assume $c = b$, i.e., $f \leq g$ on $(a, b]$. Then for any $\epsilon > 0$ since $0 \leq f(x) \leq g(x)$ for all $x \in [a+\epsilon, b]$, we have $\int_{a+\epsilon}^b f(x)\,\mathrm{d}x \leq \int_{a+\epsilon}^b g(x)\,\mathrm{d}x$.

Hence, (i) when $\int_a^b g(x)\,\mathrm{d}x$ is convergent, then $\lim_{\epsilon\to 0+}\int_{a+\epsilon}^b f(x)\,\mathrm{d}x \le \lim_{\epsilon\to 0+}\int_{a+\epsilon}^b g(x)\,\mathrm{d}x < \infty$. So $\int_{a+\epsilon}^b f(x)\,\mathrm{d}x$ is convergent.

(ii) On the other hand, when $\int_a^b f(x)\,\mathrm{d}x$ is divergent then $\lim_{\epsilon\to 0+}\int_{a+\epsilon}^b f(x)\,\mathrm{d}x = \infty$. This implies that $\lim_{\epsilon\to 0+}\int_{a+\epsilon}^b g(x)\,\mathrm{d}x = \infty$. Thus $\int_a^b g(x)\,\mathrm{d}x$ is divergent. ■

In a similar way one can prove the following theorem and hence I omit it and ask the students to write the proof as an exercise.

THEOREM. 2.8 (Comparison test:) *If f, g are integrable over $[a, X)$ for all $X \ge a$ and $0 \le f(x) \le g(x)$ for all $x \in [a,\infty)$ then (i) if $\int_a^\infty g(x)\,\mathrm{d}x$ is convergent then so is $\int_a^\infty f(x)\,\mathrm{d}x$ and (ii) if $\int_a^\infty f(x)\,\mathrm{d}x$ is divergent then so is $\int_a^\infty f(x)\,\mathrm{d}x$.*

THEOREM. 2.9 *If the functions $f, g : (a, b] \to \mathbb{R}$ have the only point of infinite discontinuity at $x = a$, both f, g are positive on $(a, b]$ such that $\lim_{x\to a+}\frac{f(x)}{g(x)} = l$, where $0 < l < \infty$, then the integrals $\int_a^b f(x)\,\mathrm{d}x$ and $\int_a^b g(x)\,\mathrm{d}x$ either both converge or both diverge.*

PROOF. Since both f, g are positive, $l > 0$. Choose $\epsilon = \frac{l}{2}$. Then there exists $\delta > 0$ such that $a + \delta < b$ and $|\frac{f(x)}{g(x)} - l| < \epsilon$ for all $x \in (a, a+\delta)$, i.e., $l - \epsilon < \frac{f(x)}{g(x)} < l + \epsilon$ for all $x \in (a, a+\delta)$. Since $g > 0$ and $\epsilon = \frac{l}{2}$, we have

$$\frac{l}{2}\cdot g(x) < f(x) < \frac{3l}{2}\cdot g(x) \quad \text{for all} \quad x \in (a, a+\delta).$$

Put $c = a + \delta$. Assume that the integral $\int_a^b f(x)\,\mathrm{d}x$ is convergent. This implies that

$$\begin{aligned}\int_a^c f(x)\,\mathrm{d}x \ \text{ is convergent} \quad &\Rightarrow \int_a^c \frac{l}{2}\, g(x)\,\mathrm{d}x \ \text{ is convergent}\\ &\Rightarrow \int_a^c g(x)\,\mathrm{d}x \ \text{ is convergent}\\ &\Rightarrow \int_a^c g(x)\,\mathrm{d}x + \int_c^b g(x)\,\mathrm{d}x \ \text{ is convergent}\\ &\Rightarrow \int_a^b g(x)\,\mathrm{d}x \ \text{ is convergent.}\end{aligned}$$

Also assuming $\int_a^b f(x)\,\mathrm{d}x$ is divergent, since $\int_c^b f(x)\,\mathrm{d}x$ is a proper integral, we have

$$\int_a^c f(x)\,\mathrm{d}x \ \text{ is divergent} \quad \Rightarrow \quad \int_a^c \frac{3l}{2}\, g(x)\,\mathrm{d}x \ \text{ is divergent}$$

$$\Rightarrow \int_a^c g(x)\,\mathrm{d}x \text{ is divergent}$$
$$\Rightarrow \int_a^c g(x)\,\mathrm{d}x + \int_c^b g(x)\,\mathrm{d}x \text{ is divergent}$$
$$\Rightarrow \int_a^b g(x)\,\mathrm{d}x \text{ is divergent.}$$

Hence either both the integrals are convergent or both are divergent. ■

The following theorem is stated without proof and the students are asked to write it by following the method adopted in the above one.

THEOREM. 2.10 *If the functions $f, g : [a, b) \to \mathbb{R}$ have the only point of infinite discontinuity at $x = b$, both f, g are positive on $[a, b)$ such that $\lim_{x\to b-} \frac{f(x)}{g(x)} = l$, where $0 < l < \infty$, then the integrals $\int_a^b f(x)\,\mathrm{d}x$ and $\int_a^b g(x)\,\mathrm{d}x$ either both converge or both diverge.*

Analogous results are valid for the integrals when the range of integration is infinite and the integrand has no infinite discontinuity.

THEOREM. 2.11 *Assume that $f, g : [a, \infty) \to \mathbb{R}$ are positive and has no infinite discontinuity on its domain, also $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$, where $0 < l < \infty$. Then the integrals $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_a^\infty g(x)\,\mathrm{d}x$ are either both convergent or both divergent.*

PROOF. Since $l > 0$, choose $\epsilon > 0$ such that $l - \epsilon > 0$. For this ϵ there exists $m > a$ such that $|\frac{f(x)}{g(x)} - l| < \epsilon$ whenever $x > m$. This implies that $l - \epsilon < \frac{f(x)}{g(x)} < l + \epsilon$ whenever $x > m$, i.e., $(l - \epsilon)g(x) < f(x) < (l + \epsilon)g(x)$ whenever $x > m$. Since $\int_a^\infty f(x)\,\mathrm{d}x = \int_a^m f(x)\,\mathrm{d}x + \int_m^\infty f(x)\,\mathrm{d}x$ and $\int_a^m f(x)\,\mathrm{d}x$ is a proper integral, we have

$$\begin{aligned}\int_a^\infty f(x)\,\mathrm{d}x \text{ is convergent} \quad &\Rightarrow \int_m^\infty f(x)\,\mathrm{d}x \text{ is convergent}\\ &\Rightarrow \int_m^\infty (l-\epsilon)g(x)\,\mathrm{d}x \text{ is convergent}\\ &\Rightarrow \int_m^\infty g(x)\,\mathrm{d}x \text{ is convergent}\\ &\Rightarrow \int_a^\infty g(x)\,\mathrm{d}x \text{ is convergent.}\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_a^\infty f(x)\,\mathrm{d}x \text{ is divergent} \quad &\Rightarrow \int_m^\infty f(x)\,\mathrm{d}x \text{ is divergent}\\ &\Rightarrow \int_m^\infty (l+\epsilon)g(x)\,\mathrm{d}x \text{ is divergent}\end{aligned}$$

$$\Rightarrow \int_m^\infty g(x)\,\mathrm{d}x \ \text{ is divergent}$$
$$\Rightarrow \int_a^\infty g(x)\,\mathrm{d}x \ \text{ is divergent.}$$

Thus, either both are convergent or both are divergent. ■

Analogously one can prove that

THEOREM. 2.12 *Assume that $f, g : (-\infty, b] \to \mathbb{R}$ are positive and has no infinite discontinuity on its domain, also $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = l$, where $0 < l < \infty$. Then the integrals $\int_{-\infty}^b f(x)\,\mathrm{d}x$ and $\int_{-\infty}^b g(x)\,\mathrm{d}x$ are either both convergent or both divergent.*

We omit its proof, interested students can do it as an exercise.

EXAMPLE. 2.13 Test the convergence of the integral $\displaystyle\int_0^1 \frac{\mathrm{d}x}{x^{\frac{3}{2}}(1+x^2)^{\frac{5}{2}}}$.

Here $f(x) = \frac{1}{x^{\frac{3}{2}}(1+x^2)^{\frac{5}{2}}}$ has only infinite discontinuity at $x = 0$. Let us take $g(x) = \frac{1}{x^{\frac{3}{2}}}, 0 < x \leq 1$. Then $\displaystyle\lim_{x\to0} \frac{f(x)}{g(x)} = \lim_{x\to0} \frac{1}{(1+x^2)^{\frac{5}{2}}} = 1 < \infty$. Hence both the integrals $\int_0^1 f(x)\,\mathrm{d}x$ and $\int_0^1 g(x)\,\mathrm{d}x$ have the same convergence behaviour. Since the integral $\displaystyle\int_0^1 \frac{dx}{x^{\frac{3}{2}}}$ is divergent, $(n = \frac{3}{2} > 1)$, the integral $\displaystyle\int_0^1 f(x)\,\mathrm{d}x$ is divergent.

EXAMPLE. 2.14 Test the convergence of the integral $\displaystyle\int_0^\infty \frac{x\,\mathrm{d}x}{(1+x^2)^3}$.

Here $f(x) = \frac{x}{(1+x^2)^3}$ has no infinite discontinuity in $[0, \infty)$. Counting the degrees in the numerator and denominator of f we take $g(x) = \frac{1}{x^5}, x > 0$. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{(1+x^2)^3}\frac{x^5}{1} = \lim_{x\to\infty} \frac{x^6}{(1+x^2)^3} = \lim_{x\to\infty} \frac{1}{(\frac{1}{x^2}+1)^3} = 1 < \infty.$$

Since the integral $\displaystyle\int_0^\infty \frac{1}{x^5}\,\mathrm{d}x$ is convergent $(p = 5 > 1)$ it follows that $\displaystyle\int_0^1 f(x)\,\mathrm{d}x$ is convergent.

2.4 Beta and Gamma Function

In this section we deal with two improper integrals which have much importance in various applications of mathematics. They are known as Beta Functions and Gamma Functions.

2.4.1 Beta Function:

The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$ is known as *beta function* and is denoted by $\beta(m,n)$.

THEOREM. 2.15 *The beta function* $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$ *is convergent when* $m > 0$ *and* $n > 0$.

PROOF. It is obvious that the integral is proper when both $m, n \geq 1$. So we have to check when $m < 1$ or $n < 1$ or both. We divide the integral as

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,\mathrm{d}x + \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x.$$

When $m < 1$ the first integrand has an infinite discontinuity at $x = 0$ and when $n < 1$ the second integrand has an infinite discontinuity at $x = 1$. Let $f(x) = x^{m-1}(1-x)^{n-1}, 0 < x < 1$.

When $m < 1$: take $g(x) = x^{m-1}$. Then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{x^{m-1}(1-x)^{n-1}}{x^{m-1}} = \lim_{x\to 0+} (1-x)^{n-1} = 1$.

Now, $\int_0^{\frac{1}{2}} g(x)\,\mathrm{d}x = \int_0^{\frac{1}{2}} x^{m-1}\,\mathrm{d}x = \int_0^{\frac{1}{2}} \frac{1}{x^{1-m}}\,\mathrm{d}x$ is convergent if $1 - m < 1$, i.e., if $m > 0$.

Hence the integral $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$ is convergent when $m > 0$.

When $n < 1$: take $h(x) = (1-x)^{n-1}$. Then $\lim_{x\to 1-} \frac{f(x)}{h(x)} = \lim_{x\to 1-} \frac{x^{m-1}(1-x)^{n-1}}{(1-x)^{n-1}} = \lim_{x\to 1-} x^{m-1} = 1$.

Now, $\int_{\frac{1}{2}}^1 h(x)\,\mathrm{d}x = \int_{\frac{1}{2}}^1 (1-x)^{n-1}\,\mathrm{d}x = \int_{\frac{1}{2}}^1 \frac{1}{(1-x)^{1-n}}\,\mathrm{d}x$ is convergent if $1 - n < 1$, i.e., if $n > 0$. Hence the integral $\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$ is convergent when $n > 0$.

From the above two, the integral $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$ is convergent if $m > 0$ and $n > 0$. ■

Henceforth, whenever we write $\beta(m,n)$ we shall assume $m > 0$ and $n > 0$, unless stated otherwise.

2.4.2 Properties of Beta Functions

1. $\beta(m,n) = \beta(n,m)$.

   Putting $x = 1 - y$, $dx = -dy$, when $x \to 0, y \to 1$ and when $x \to 1, y \to 0$, we have,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x = \int_1^0 (1-y)^{m-1}y^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}dy \;=\; \beta(n,m).$$

2. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \ \cos^{2n-1}\theta \ d\theta.$

   Substituting $x = \sin^2\theta$, $\mathrm{d}x = 2\sin\theta\cos\theta \ d\theta$. When $x \to 0$ then $\theta \to 0$, when $x \to 1$, $\theta \to \pi/2$. So,

$$\begin{aligned}\beta(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x \\ &= \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta \ d\theta \\ &= 2\int_0^{\pi/2} \sin^{2m-1}\theta \ \cos^{2n-1}\theta \ d\theta.\end{aligned}$$

3. $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,\mathrm{d}x.$

   Take a substitution $x = \frac{y}{1+y}$. Then $dx = \frac{1}{(1+y)^2}\,dy$ and $1-x = \frac{1}{1+y}$. Also $x = \frac{y}{1+y}$ gives $y = \frac{x}{1-x}$, hence when $x \to 0, y \to 0$ and when $x \to 1, y \to \infty$. Thus the integral becomes,

$$\begin{aligned}\beta(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x \\ &= \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1}\left(\frac{1}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}\,dy \\ &= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy.\end{aligned}$$

4. For $k>0$, $\beta(m,n) = k\int_0^1 x^{mk-1}(1-x^k)^{n-1}\,\mathrm{d}x.$

   This can be proved by substituting $x = y^k, k > 0$. The students are required to do it.

5. $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1).$

$$\begin{aligned}\beta(m,n+1) &= \int_0^1 x^{m-1}(1-x)^n\,\mathrm{d}x \;=\; \int_0^1 x^{m-1}(1-x)^{n-1}(1-x)\,\mathrm{d}x \\ &= \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x - \int_0^1 x^m(1-x)^{n-1}\,\mathrm{d}x \\ &= \beta(m,n) - \beta(m+1,n).\end{aligned}$$

   Hence the result.

EXAMPLE. 2.16 Evaluate $\beta(\frac{1}{2},\frac{1}{2})$.

$$\beta(\tfrac{1}{2},\tfrac{1}{2}) = 2\int_0^{\pi/2} \sin^{2\cdot\frac{1}{2}-1}\theta \ \cos^{2\cdot\frac{1}{2}-1}\theta \ d\theta \;=\; 2\int_0^{\pi/2} d\theta \;=\; 2\times\frac{\pi}{2} \;=\; \pi.$$

Problem. 2.17 Prove that 1. $\beta(m, n+1) = \frac{m}{m+n} \cdot \beta(m, n)$ and 2. $\beta(m+1, n) = \frac{n}{m+n} \cdot \beta(m, n)$.

2.4.3 Gamma Function:

The improper integral $\int_0^\infty e^{-x} x^{n-1}\, dx$ is known as *Gamma Function* and is denoted by $\Gamma(n)$. Thus,

Theorem. 2.18 *The Gamma Function* $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$ *converges if* $n > 0$.

Proof. The function $f(x) = e^{-x}x^{n-1}$ has an infinite discontinuity at $x = 0$ when $n < 1$. So we have to check convergence both at $x = 0$ and at ∞. We write the integral as,

$$\Gamma(n) \quad = \quad \int_0^\infty e^{-x} x^{n-1}\, dx \;=\; \int_0^1 e^{-x} x^{n-1}\, dx + \int_1^\infty e^{-x} x^{n-1}\, dx.$$

To check convergence at 0, take $g(x) = x^{n-1}$. Then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{e^{-x}x^{n-1}}{x^{n-1}} = \lim_{x\to 0+} e^{-x}$ $= 1 < \infty$. Also the integral $\int_0^1 g(x)\, dx = \int_0^1 x^{n-1}\, dx = \int_0^1 \frac{dx}{x^{1-n}}$ is convergent if $1 - n < 1$, i.e., if $n > 0$. Hence the integral $\int_0^1 e^{-x}x^{n-1}\, dx$ is convergent if $n > 0$.

We check the convergence at ∞ a little elaborately, in a few steps.

1. First note that $\int_1^\infty e^{-kx}\, dx$ is convergent for any $k > 0$. One can easily verify this from definition.

2. For any positive integer n, there exists $M > 0$ such that $e^{-x}x^{n-1} < e^{-\frac{1}{2}x}$ for all $x > M$. To verify this evaluate the limit $\lim_{x\to\infty} \frac{x^{n-1}}{e^{\frac{1}{2}x}} = 0$ [by using L' Hospital's rule ($\frac{\infty}{\infty}$) form]. Hence for taking $\epsilon = 1$ there exists $M > 0$ such that $|\frac{x^{n-1}}{e^{\frac{1}{2}x}} - 0| < 1$ whenever $x > M$, i.e., $x^{n-1} < e^{\frac{1}{2}x}$ for all $x > M$. Multiplying both sides by e^{-x} we have $e^{-x}x^{n-1} < e^{-\frac{1}{2}x}$ for all $x > M$. By comparison and using item 1 above, we have $\int_1^\infty e^{-x}x^{n-1}\, dx$ is convergent whenever n is a positive integer.

3. When n is a real number greater than 1 then $[n]$ is a positive integer and $e^{-x}x^{n-1} < e^{-x}x^{[n]}$. Since the integral $\int_1^\infty e^{-x}x^{[n]}\, dx$ is convergent, by comparison the integral $\int_1^\infty e^{-x}x^{n-1}\, dx$ is convergent for any real number $n > 1$.

4. When $0 < n < 1$, since we have $1 \le x^{n-1} \le x$ for all $x > 1$, we get $\frac{1}{e^{\frac{1}{2}x}} \le \frac{x^{n-1}}{e^{\frac{1}{2}x}} \le \frac{x}{e^{\frac{1}{2}x}}$ for all $x > 1$. Since $\lim_{x\to\infty} \frac{1}{e^{\frac{1}{2}x}} = 0 = \lim_{x\to\infty} \frac{x}{e^{\frac{1}{2}x}}$, by sandwich rule we

have $\lim_{x\to\infty} \frac{x^{n-1}}{e^{\frac{1}{2}x}} = 0$. By the method similar to that adopted in item 2 we can prove $\int_1^\infty e^{-x}x^{n-1}\,\mathrm{d}x$ is convergent when $0 < n < 1$.

Hence the integral $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,\mathrm{d}x$ is convergent if $n > 0$. ■

Henceforth, whenever we write $\Gamma(n)$, we shall assume that $n > 0$, unless we state otherwise.

2.4.4 Properties of Gamma Function

1. $\Gamma(n+1) = n\Gamma(n)$.

$$\begin{aligned}
\text{We have } \Gamma(n+1) &= \int_0^\infty e^{-x}x^n\,\mathrm{d}x \;=\; \lim_{X\to\infty}\int_0^X e^{-x}x^n\,\mathrm{d}x \\
&= \lim_{X\to\infty}\left\{\left[x^n(-e^{-x})\right]_0^X + n\int_0^X e^{-x}x^{n-1}\,\mathrm{d}x\right\} \\
&= \lim_{X\to\infty}\left[-X^n e^{-X} + 0\right] + n\lim_{X\to\infty}\int_0^X e^{-x}x^{n-1}\,\mathrm{d}x \\
&= \lim_{X\to\infty}\left[-\frac{X^n}{e^X}\right] + n\int_0^\infty e^{-x}x^{n-1}\,\mathrm{d}x \;=\; 0 + n\Gamma(n).
\end{aligned}$$

2. For a positive integer n, $\Gamma(n+1) = n!$.

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) \\
\Gamma(n) &= (n-1)\Gamma(n-1) \\
\vdots \quad & \quad \vdots \\
\Gamma(3) &= 2\Gamma(2) \\
\Gamma(2) &= 1\Gamma(1)
\end{aligned}$$

Thus $\Gamma(n+1) = n(n-1)(n-2)\cdots 3{\cdot}2{\cdot}1{\cdot}\Gamma(1) = n!{\cdot}\Gamma(1)$. Since $\Gamma(1) = \int_0^\infty e^{-x}x^{1-1}\,\mathrm{d}x = \lim_{X\to\infty}\int_0^X e^{-x}\,\mathrm{d}x = \lim_{X\to\infty}\left[-e^{-x}\right]_0^X = \lim_{X\to\infty}\left[-\frac{1}{e^X}+1\right] = 1$, it follows that Thus $\Gamma(n+1) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$.

3. For $k > 0$, $\Gamma(n) = k\int_0^\infty e^{-x^k}x^{kn-1}\,\mathrm{d}x$.

Put $x = y^k, k > 0$. Then when $x \to 0, y \to 0$ and when $x \to \infty, y \to \infty$. Also $dx = ky^{k-1}\,dy$. After this substitution the integral becomes,

$$\begin{aligned}
\Gamma(n) &= \int_0^\infty e^{-x}x^{n-1}\,\mathrm{d}x \;=\; \int_0^\infty e^{-y^k}(y^k)^{n-1}ky^{k-1}\,dy \\
&= k\int_0^\infty e^{-y^k}y^{kn-1}\,dy.
\end{aligned}$$

Hence the result.

4. For $0 < n < 1$, $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$.

   Proof of this result is omitted as it involves topics beyond the curriculum.

2.4.5 Relation between Beta Function and Gamma Function

THEOREM. 2.19 *For $m, n > 0$, $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.*

PROOF. We know for $k > 0$, $\Gamma(n) = k\displaystyle\int_0^\infty e^{-x^k} x^{kn-1}\,\mathrm{d}x$. Hence taking $k = 2$ we have

$$\begin{aligned}\Gamma(n)\Gamma(m) &= 2\int_0^\infty e^{-x^2} x^{2n-1}\,\mathrm{d}x \cdot 2\int_0^\infty e^{-y^2} y^{2m-1}\,dy \\ &= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2n-1} y^{2m-1}\,\mathrm{d}x\,dy\end{aligned}$$

Take $x = r\cos\theta$ and $y = r\sin\theta$, $0 < r < \infty, 0 \le \theta \le \frac{\pi}{2}$ and $\frac{\partial(x,y)}{\partial(r,\theta)} = r$, the integral becomes,

$$\begin{aligned}\Gamma(n)\Gamma(m) &= 4\int_{r=0}^\infty \int_{\theta=0}^{\frac{\pi}{2}} e^{-r^2} r^{2n-1}\cos^{2n-1}\theta\; r^{2m-1}\sin^{2m-1}\theta\; r d\theta dr \\ &= 4\int_{r=0}^\infty \int_{\theta=0}^{\frac{\pi}{2}} e^{-r^2} r^{2(n+m)-1}\cos^{2n-1}\theta\;\sin^{2m-1}\theta\; d\theta dr \\ &= 2\int_0^\infty e^{-r^2} r^{2(n+m)-1}\,dr \cdot 2\int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta\;\sin^{2m-1}\theta\; d\theta \\ &= \Gamma(n+m)\cdot\beta(n,m).\end{aligned}$$

Hence $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

EXAMPLE. 2.20 1. Find $\Gamma(\frac{1}{2})$.

From the relation between beta and gamma functions it follows that

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left(\Gamma(\frac{1}{2})\right)^2}{\Gamma(1)} = \left(\Gamma(\frac{1}{2})\right)^2.$$

Since $\beta(\frac{1}{2},\frac{1}{2}) = \pi$, therefore, $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi$. Hence $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

2. Establish the relation $\displaystyle\int_0^{\pi/2} \sin^p\theta\;\cos^q\theta\; d\theta = \frac{\beta(\frac{p+1}{2},\frac{q+1}{2})}{2} = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})}$, $p, q > -1$.

$$\text{We know, } \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\;\cos^{2n-1}\theta\; d\theta.$$

Putting $2m-1=p, 2n-1=q$, $m=\frac{p+1}{2}, n=\frac{q+1}{2}$. When $m,n>0$, then $p>-1, q>-1$. So we get

$$2\int_0^{\pi/2} \sin^p\theta\ \cos^q\theta\ d\theta = \beta(\frac{p+1}{2}, \frac{q+1}{2}),$$

Again, since $\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$,

$$\beta(\frac{p+1}{2}, \frac{q+1}{2}) = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+1}{2}+\frac{q+1}{2})} = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q+2}{2})}.$$

Thus,

$$\int_0^{\pi/2} \sin^p\theta\ \cos^q\theta\ d\theta = \frac{1}{2}\cdot\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q+2}{2})}.$$

3. Evaluate $\displaystyle\int_0^{\pi/2} \sin^4\theta\ \cos^6\theta\ d\theta$.

$$\begin{aligned}\int_0^{\pi/2} \sin^4\theta\ \cos^6\theta\ d\theta &= \frac{\Gamma(\frac{4+1}{2})\ \Gamma(\frac{6+1}{2})}{2\Gamma(\frac{4+6+2}{2})} = \frac{\Gamma(\frac{5}{2})\ \Gamma(\frac{7}{2})}{2\Gamma(6)} \\ &= \frac{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})\ \frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{2\times 5!} \\ &= \frac{\frac{45}{32}\left[\Gamma(\frac{1}{2})\right]^2}{2\times 120} = \frac{45}{64\times 120}\pi = \frac{3}{64\times 8}\pi = \frac{3\pi}{512}.\end{aligned}$$

2.4.6 Problems

1. Prove that (i) $\beta(m,1)=\frac{1}{m}$, (ii) $\Gamma(5/2)=\frac{3}{4}\sqrt{\pi}$, (iii) $\Gamma(6)=120$.

2. Evaluate the following integrals using beta and gamma functions:

$$(i)\ \int_0^1 x^3(1-x^2)^{5/2}\,\mathrm{d}x \quad (ii)\ \int_0^1 x^4(1-x^2)^3\,\mathrm{d}x \quad (iii)\ \int_0^1 x^{3/2}(1-x)^{3/2}\,\mathrm{d}x$$

$$(iv)\ \int_0^1 x^{5/2}(1-x)\,\mathrm{d}x \qquad (v)\ \int_0^{\pi/2} \cos^4 x\,\mathrm{d}x \qquad (vi)\ \int_0^{\pi/2} \sin^6 x\cos^3 x\,\mathrm{d}x$$

Ans: (i) $\frac{2}{63}$ (ii) $\frac{16}{1155}$ (iii) $\frac{3\pi}{128}$ (iv) $\frac{4}{63}$ (v) $\frac{3\pi}{16}$ (vi) $\frac{2}{63}$.

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