Study Material on CC-13

Department of Mathematics, P. R. Thakur Govt. College
MTMACOR13T: (Semester - 6)

Syllabus:

Unit-1 : Metric spaces: Definition and examples. Open and closed balls, neighbourhood, open set, interior of a set. Limit point of a set, closed set, diameter of a set, subspaces, dense sets, separable spaces. Sequences in Metric Spaces, Cauchy sequences. Complete Metric Spaces, Cantor's theorem.

Unit 2 : Continuous mappings, sequential criterion and other characterizations of continuity, Uniform continuity, Connectedness, connected subsets of R. Compactness: Sequential compactness, Heine-Borel property, Totally bounded spaces, finite intersection property, and continuous functions on compact sets. Homeomorphism, Contraction mappings, Banach Fixed point Theorem and its application to ordinary differential equation.

1 Metric space and related concepts

Metric space is the generalisation of the Euclidean spaces $\mathbb{R}^n$, where the distance function plays the crucial role to most of the concepts regarding continuity, convergence etc.

1.1 Basic Definitions, Open Sets and Closed Sets

Definition. 1.1 Let X be a non-empty set and $\rho : X \times X \to \mathbb{R}$ be a function. ρ is said to be a *metric* on X if the following conditions hold:

M1: For all x, y in X, $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$.

M2: For all x, y in X, $\rho(x, y) = \rho(y, x)$.

M3: For all x, y, z in X, $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ — this rule is known as triangular inequality.

If ρ is a metric on a set X then the pair (X, ρ) is called a *metric space*.

Example. 1.2

1. Let $\mathbb{R}$ be the set of reals. We define $\rho(x, y) = |x - y|\ \forall x, y \in \mathbb{R}$, then ρ is a metric on $\mathbb{R}$, called the *usual metric on* $\mathbb{R}$ and hence $(\mathbb{R}, \rho)$ is a metric space.
2. Let $\mathbb{C}$ be the set of complex numbers. If we define $\rho(z_1, z_2) = |z_1 - z_2|\ \ \forall z_1, z_2 \in \mathbb{C}$, then $(\mathbb{C}, \rho)$ is a metric space.

3. Let $\mathbb{R}^n$ be the set of all n-tuples of reals. If we define

$$\rho(x,y) = \sqrt{\sum (x_i - y_i)^2}, \text{ for all } x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n,$$

then ρ is a metric on $\mathbb{R}^n$, called the *Euclidean metric* on $\mathbb{R}^n$ and hence $(\mathbb{R}^n, \rho)$ is a metric space. This metric space is called the *n-dimensional Euclidean space.*

4. Let $B(X)$ be the set of all real valued bounded functions on a non-empty set set X. If we define

$$\rho(f,g) = \sup\{|f(x) - g(x)| : x \in X\} \quad \text{for all } f, g \in B(X),$$

then ρ is a metric on $B(X)$, called the *supnorm metric* on $B(X)$ and hence $(B(X), \rho)$ is a metric space.

5. Let X be any non-empty set. We define $\rho : X \times X \to \mathbb{R}$ by

$$\begin{aligned} \rho(x,y) &= 0, \text{ if } x = y \\ &= 1, \text{ if } x \neq y. \end{aligned}$$

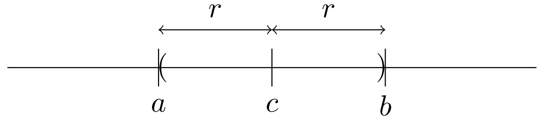
Then ρ is a metric on X. Such a metric space (X, ρ) is called a *discrete metric space.*

Definition. 1.3 Let (X, ρ) be a metric space and $a \in X$. Then the set

$$\{x \in X : \rho(a,x) < r\},$$

where r is a positive real number, is called an *open ball* or *open sphere* with center at a and of radius r and is denoted by $S_r(a, \rho)$ or simply by $S_r(a)$ when no confusion about ρ is likely to arise. The notations $B_\rho(a, r)$ or $B(a, r)$ are also used to denote an open ball with center at a and radius r.

Example. 1.4 1. In $\mathbb{R}$, the open sphere with center at c and radius r is $S_r(c) = B(c, r) = \{x \in \mathbb{R} : |x - c| < r\} = (c - r, c + r)$ which is a bounded open interval in $\mathbb{R}$. Also any bounded open interval (a, b) can be written as $(a, b) = S_r(c)$ where $r = \frac{b-a}{2}$, the half of the length of the interval, and $c = \frac{a+b}{2}$, the mid-point of the interval. Thus the open spheres in $\mathbb{R}$ are exactly the bounded open intervals in $\mathbb{R}$.



2. In Euclidean plane $\mathbb{R}^2$, for any $(a, b) \in \mathbb{R}^2$ and for any $r > 0$,

$$\begin{aligned} S_r((a,b)) &= \{(x,y) \in \mathbb{R}^2 : \|(x,y) - (a,b)\| < r\} \\ &= \{(x,y) \in \mathbb{R}^2 : \sqrt{(x-a)^2 + (y-b)^2} < r\} \\ &= \{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 < r^2\} \end{aligned}$$

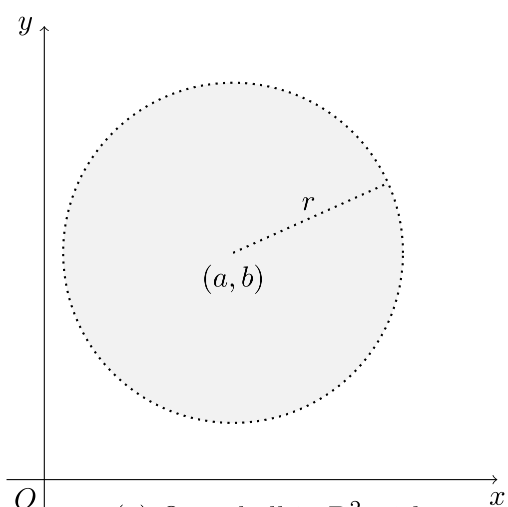
which is nothing but the the set of points lying inside the circle whose centre is at (a, b) and radius is r.

3. For a non-empty set X consider the metric space $B(X)$. For $f \in B(X)$ and $r > 0$,

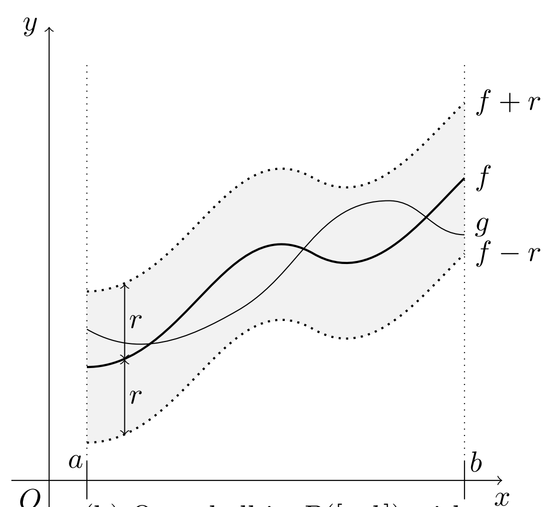
$$\begin{aligned} S_r(f) &= \{g \in B(X) : \sup\{|f(x) - g(x)| : x \in X\} < r\} \\ &= \{g \in B(X) : |f(x) - g(x)| < r \ \forall x \in X\} \\ &= \{g \in B(X) : f(x) - r < g(x) < f(x) + r \ \forall x \in X\}. \end{aligned}$$

   Thus, $S_r(f)$ is the set of all those members g of $B(X)$ whose *graph* lies between those of $f - r$ and $f + r$.

4. Let X be a non-empty set equipped with discrete metric ρ. It is easy to verify that for $a \in X$ and $r > 0$, $S_r(a) = \{a\}$ if $r \leq 1$ and $S_r(a) = X$ if $r > 1$.



(a) Open ball in $\mathbb{R}^2$ with center at (a, b) and radius r



(b) Open ball in $B([a, b])$ with center at f and radius r

In the above, figure (a) is the open ball in $\mathbb{R}^2$ which is actually an open circular disk with center at (a, b) and radius r. The figure (b) is in the metric space $B([a, b])$ of the set of all bounded functions defined in the closed interval $[a, b]$ with supnorm metric, the open ball $S_r(f)$, centered at f and radius r. It considts of all the bounded functions $g : [a, b] \to \mathbb{R}$ such that $f(x) - r < g(x) < f(x) + r$, for all $x \in [a, b]$.

THEOREM. 1.5 *Let (X, ρ) be a metric space. Let $S_{r_1}(a), S_{r_2}(b)$ be two open spheres in (X, ρ). Then*

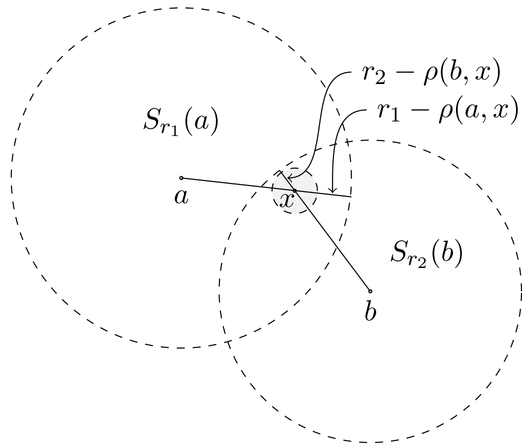
1. *For each x in $S_{r_1}(a)$ there exists $\delta_1 > 0$ such that $S_{\delta_1}(x) \subset S_{r_1}(a)$.*

2. *For each x in $S_{r_1}(a) \cap S_{r_2}(b)$ there exists $\delta_2 > 0$ such that $S_{\delta_2}(x) \subset S_{r_1}(a) \cap S_{r_2}(b)$.*

Proof. 1. If $x \in S_{r_1}(a)$ then $\rho(a,x) < r_1$. Let $\delta_1 = r_1 - \rho(a,x)$, then $\delta_1 > 0$. If $y \in S_{\delta_1}(x)$ then $\rho(x,y) < \delta_1$ and hence $\rho(a,y) \le \rho(a,x) + \rho(x,y) < \rho(a,x) + \delta_1 = r_1$. So, $y \in S_{r_1}(a)$. Thus $S_{\delta_1}(x) \subset S_{r_1}(a)$.

2. Let $x \in S_{r_1}(a) \cap S_{r_2}(b)$. Then $\rho(a,x) < r_1$ and $\rho(b,x) < r_2$. Choose

$$\delta_2 = \min\{r_1 - \rho(a,x), r_2 - \rho(b,x)\}.$$



Clearly $\delta_2 > 0$. If $y \in S_{\delta_2}(x)$ then $\rho(x,y) < \delta_2$, so

$$\begin{aligned} \rho(a,y) &\le \rho(a,x) + \rho(x,y) \\ &< \rho(a,x) + \delta_2 \\ &< \rho(a,x) + (r_1 - \rho(a,x)) \\ &= r_1. \end{aligned}$$

Hence $y \in S_{r_1}(a)$, i.e., $S_{\delta_2}(x) \subset S_{r_1}(a)$. Similarly we can show that $S_{\delta_2}(x) \subset S_{r_2}(b)$. Thus $S_{\delta_2}(x) \subset S_{r_1}(a) \cap S_{r_2}(b)$. ■

Definition. 1.6 Let (X, ρ) be a metric space, $A \subset X$ and $x \in X$. Then a is said to be an *interior point* of A if there exists $r > 0$ such that $S_r(a) \subset A$.

A set $N \subset X$ is said to be a *neighbourhood* of a point x if x is an interior point of N.

Theorem. 1.7 *Let (X, ρ) be a metric space, for $x \in X$ we denote by $\mathcal{N}(x, \rho)$ or simply by $\mathcal{N}_x$ the set of all neighbourhoods of x. Then For all $x \in X$,*

1. *$\mathcal{N}_x \neq \emptyset$ and $x \in N$ for each $N \in \mathcal{N}_x$.*
2. *For all $A, B \subset X$, $A \supset B$ and $B \in \mathcal{N}_x$ implies that $A \in \mathcal{N}_x$.*
3. *For all $A, B \subset X$, $A, B \in \mathcal{N}_x$ implies that $A \cap B \in \mathcal{N}_x$.*
4. *If $A \in \mathcal{N}_x$ then there exists $B \in \mathcal{N}_x$ such that $B \subset A$ and $B \in \mathcal{N}_y$ for all $y \in B$.*
5. *If $A \in \mathcal{N}_x$ then there exists $B \in \mathcal{N}_x$ such that $A \in \mathcal{N}_y$ for all $y \in B$.*

Proof. 1. For any $x \in X, r > 0$, $S_r(x) \subset X$ and hence $X \in \mathcal{N}_x$. So $\mathcal{N}_x \neq \emptyset$. Also of $N \in \mathcal{N}_x$ then there exists $r > 0$ such that $S_r(x) \subset N$ and hence $x \in N$.

2. If $B \in \mathcal{N}_x$ then there exists $r > 0$ such that $S_r(x) \subset B$. Since $B \subset A$, $S_r(x) \subset A$, hence $A \in \mathcal{N}_x$.

3. $A, B \in \mathcal{N}_x$ implies that there exist $r_1, r_2 > 0$ such that $S_{r_1}(x) \subset A$ and $S_{r_2}(x) \subset B$. Let $r = \min\{r_1, r_2\}$. Then $S_r(x) \subset A \cap B$ and hence $A \cap B \in \mathcal{N}_x$.

4. Let $A \subset N_x$, then there exists $r > 0$ such that $S_r(x) \subset A$. Put $B = S_r(x)$. So $B \in \mathcal{N}_x$ and $B \subset A$. If $y \in B$ we can find $r_1 > 0$ such that $S_{r_1}(y) \subset B$ (we take $r_1 = r - \rho(x, y)$). Thus $B \in \mathcal{N}(y)$.

5. Follows from 2 and 4. ■

DEFINITION. 1.8 Let (X, ρ) be a metric space, a set $V \subset X$ is said to be an *open set* if it is a neighbourhood of each of its points.

Every open sphere in a metric space is open set (Theorem 1.5(1))

THEOREM. 1.9 *Let (X, ρ) be a metric space and $\mathcal{T}_\rho$ denote the set of all open sets of (X, ρ). Then*

1. *$\emptyset, X \in \mathcal{T}_\rho$.*
2. *If $V_1, V_2 \in \mathcal{T}_\rho$ then $V_1 \cap V_2 \in \mathcal{T}_\rho$.*
3. *If $\{V_i : i \in I\} \subset \mathcal{T}_\rho$ then $\cup\{V_i : i \in I\} \in \mathcal{T}_\rho$.*

PROOF. 1. Clearly X is a neighbourhood of each of its points. Also it is vacuously true that the empty set is a neighbourhood of each of its points. Thus $X, \emptyset \in \mathcal{T}_\rho$.

2. Let $a \in V_1 \cap V_2$, i.e. $a \in V_1$ and $a \in V_2$. Then $V_1 \in \mathcal{N}_a$ and $V_2 \in \mathcal{N}_a$ and hence by Theorem 1.7, $V_1 \cap V_2 \in \mathcal{N}_a$. Since a has been chosen arbitrarily in $V_1 \cap V_2$, it follows that $V_1 \cap V_2$ is a neighbourhood of each of its points. Thus $V_1 \cap V_2 \in \mathcal{T}_\rho$.

3. Let $\{V_i : i \in I\}$ be a subfamily of $\mathcal{T}_\rho$ and $a \in \cup\{V_i : i \in I\}$. Then there exists $i_0 \in I$ such that $a \in V_{i_0}$. Since $V_{i_0} \in \mathcal{T}_\rho$ and $a \in V_{i_0}$ it follows that $V_{i_0} \in \mathcal{N}_a$. Again since $V_{i_0} \subset \cup\{V_i : i \in I\}$ it follows that $\cup\{V_i : i \in I\} \in \mathcal{N}_a$. Hence $\cup\{V_i : i \in I\} \in \mathcal{T}_\rho$. ■

REMARK. 1.10 Intersection of an arbitrary collection of open sets need not be open.

EXAMPLE. 1.11 For all $n \in \mathbb{N}$ let I_n denote the open interval $(-\frac{1}{n}, \frac{1}{n})$. Then each I_n is an open sphere (see Example 1.4 (1)) and hence is an open set in $\mathbb{R}$. Note that $\cap\{I_n : n \in \mathbb{N}\} = \{0\}$, which is not an open set. Thus even a countable intersection of open sets may not be an open set.

DEFINITION. 1.12 Let (X, ρ) be a metric space, $A \subset X$. The set of all the interior points of A is said to be the *interior* of A and is denoted by A°.

THEOREM. 1.13 *Let (X, ρ) be a metric space, $A \subset X$, $x \in A$. Then*

1. *$x \in A^\circ$ if and only if there exists an open set $V \subset X$ such that $x \in V \subset A$.*
2. *A is open set if and only if $A = A^\circ$.*

PROOF. Follows from definition. ■

THEOREM. 1.14 *Let (X, ρ) be a metric space. Then*

1. *$X^\circ = X$.*
2. *For $A \subset X$, $A^\circ \subset A$.*
3. *If $A \subset B \subset X$ then $A^\circ \subset B^\circ$.*
4. *If V is an open subset of X and $A \subset X$ such that $V \subset A$ then $V \subset A^\circ$.*
5. *For $A \subset X$, $(A^\circ)^\circ = A^\circ$.*
6. *For $A, B \subset X$, $(A \cap B)^\circ = A^\circ \cap B^\circ$.*

PROOF. 1. Follows from the fact that X is open set.

2. Follows from definition of A°.

3. If $x \in A^\circ$ then there exists $r > 0$ such that $S_r(x) \subset A$. Since $A \subset B$ it follows that $S_r(x) \subset B$ and hence $x \in B^\circ$.

4. Follows from definition of interior point.

5. From above it immediately follows that $(A^\circ)^\circ \subset A^\circ$. Also, if $x \in A^\circ$ then there exists $r > 0$ such that $S_r(x) \subset A$. Since for any $y \in S_r(x)$, $A \in \mathcal{N}_y$, it follows that $S_r(x) \subset A^\circ$ and hence $x \in (A^\circ)^\circ$. Thus $A^\circ \subset (A^\circ)^\circ$.

6. Since $A \cap B \subset A$, $(A \cap B)^\circ \subset A^\circ$. Similarly $(A \cap B)^\circ \subset B^\circ$. Thus $(A \cap B)^\circ \subset A^\circ \cap B^\circ$. Also, if $x \in A^\circ \cap B^\circ$ then there exist $r_1, r_2 > 0$ such that $S_{r_1}(x) \subset A$ and $S_{r_2}(x) \subset B$. Taking $r = \min\{r_1, r_2\}$ we have $S_r(x) \subset A \cap B$ and hence $x \in (A \cap B)^\circ$. Thus $A^\circ \cap B^\circ \subset (A \cap B)^\circ$. ■

THEOREM. 1.15 *For $A \subset X$, where (X, ρ) is a metric space,*

$$A^\circ = \cup\{V \subset X : V \textit{ is open}, V \subset A\}.$$

PROOF. Let $x \in A^\circ$. Then there exists $r > 0$ such that $S_r(x) \subset A$. Let $V = S_r(x)$, then V is open set and hence $x \in \cup\{V \subset X : V \text{ is open}, V \subset A\}$, i.e.,

$$A^\circ \quad \subset \quad \cup\{V \subset X : V \text{ is open}, V \subset A\}.$$

Also, let $x \in \cup\{V \subset X : V \text{ is open }, V \subset A\}$. Then there exists open set $V \subset X$ such that $x \in V$ and $V \subset A$. Since V is open $V \subset A^\circ$ and hence $x \in A^\circ$, i.e.,

$$A^\circ \quad \supset \quad \cup\{V \subset X : V \text{ is open}, A \supset V\}.$$

Thus $A^\circ = \cup\{V \subset X : V \text{ is open}, A \supset V\}$. ■

REMARK. 1.16 From the above result one can conclude that A° is the largest (with respect to set inclusion) open set contained in A.

DEFINITION. 1.17 Let (X, ρ) be a metric space, $A \subset X$ and $a \in X$. Then a is said to be a *limit point* of A if for all $N \in \mathcal{N}_a$, $(N - \{a\}) \cap A \neq \emptyset$. The set of all the limit points of A is called the *derived set* of A and is denoted by A'.

THEOREM. 1.18 *Let A be a subset of a metric space (X, ρ), $a \in A$. Then the following statements are equivalent:*

1. *$a \in A'$.*
2. *For all $r > 0$, $(S_r(a) - \{a\}) \cap A \neq \emptyset$.*
3. *Each neighbourhood of a contains infinitely many elements of A.*

PROOF. (1) $\Rightarrow$ (2): Follows immediately since for any $r > 0$, $S_r(a)$ is a neighbourhood of a.

(2) $\Rightarrow$ (3): Assume (2) holds. If possible, suppose that there exists $N \in \mathcal{N}_a$ which contains only finitely many distinct points of A, say $x_1, x_2, \dots, x_n$. If any one of these points is a we exclude it. Let $r_i = \rho(x_i, a), 1 \leq i \leq n$. The for each i, $r_i > 0$. Also since $N \in \mathcal{N}_a$ there exists $r' > 0$ such that $S_{r'}(a) \subset N$. Put $r = \min\{r', r_1, r_2, \dots, r_n\}$. Then $r > 0$ and $S_r(a)$ contains no point of A — a contradiction to (2).

(3) $\Rightarrow$ (1): Follows from definition of limit point. ■

DEFINITION. 1.19 Let (X, ρ) be a metric space and $A \subset X$. A is said to be a *closed set* if it contains all of its limit points, i.e., if $A' \subset A$.

THEOREM. 1.20 *Let (X, ρ) be a metric space and $F \subset X$. Then F is a closed set if and only if its complement F^c in X is an open set.*

PROOF. Assume that F is a closed set. Let $x \in F^c$, then $x \notin F$. Since F is a closed set, x is not a limit point of F; hence there exists $r > 0$ such that $S_r(x) \cap F \neq \emptyset$. Thus $S_r(x) \subset F^c$ which shows that x is an interior point of F^c, i.e., $F^c \in \mathcal{N}_x$. Since x has been chosen arbitrarily in F^c, it follows that F^c is a neighbourhood of each of its points. Hence F^c is an open set.

Conversely, suppose that F^c is an open set. Let $x \notin F$, then $x \in F^c$. Since F^c is open $F^c \in \mathcal{N}_x$. Also $F \cap (F^c) = \emptyset$, which shows that $x \notin F'$. Thus $x \notin F$ implies that $x \notin F'$, hence $F' \subset F$, i.e., F is closed. ■

THEOREM. 1.21 *Let (X, ρ) be a metric space. If $\mathcal{F}_\rho$ denotes the set of all closed sets, then*

1. $X, \emptyset \in \mathcal{F}_\rho$.
2. *If* $F_1, F_2 \in \mathcal{F}_\rho$ *then* $F_1 \cup F_2 \in \mathcal{F}_\rho$.
3. *If* $\{F_i : i \in I\} \subset \mathcal{F}_\rho$ *then* $\cap\{F_i : i \in I\} \in \mathcal{F}_\rho$.

PROOF. 1. Since $\emptyset, X \in \mathcal{T}_\rho$ it follows that $X, \emptyset \in \mathcal{F}_\rho$.

2. If $F_1, F_2 \in \mathcal{F}_\rho$ then $X - F_1, X - F_2 \in \mathcal{T}_\rho$ which implies that $(X - F_1) \cap (X - F_2) \in \mathcal{T}_\rho$, i.e., $X - (F_1 \cup F_2) \in \mathcal{T}_\rho$. Hence $F_1 \cup F_2 \in \mathcal{F}_\rho$.

3. If $F_i \in \mathcal{F}_\rho$ for all $i \in I$ then $X - F_i \in \mathcal{T}_\rho$ for all $i \in I$. Hence $\cup\{X - F_i : i \in I\} \in \mathcal{T}\rho$, i.e., $(X - \cap\{F_i : i \in I\}) \in \mathcal{T}_\rho$. Thus $\cap\{F_i : i \in I\} \in \mathcal{F}_\rho$. ■

DEFINITION. 1.22 Let A be a subset of a metric space (X, ρ). Then the set $A \cup A'$ is called the *closure* of A in X and is denoted by $\bar{A}$ or by $cl(A)$.

THEOREM. 1.23 *A subset A of a metric space (X, ρ) is closed if and only if $A = \bar{A}$.*

PROOF. Assume A is a closed set. Then $A' \subset A$ and hence $\bar{A} = A \cup A' = A$. On the other hand, $\bar{A} = A \Rightarrow A \cup A' = A \Rightarrow A' \subset A$. Hence A is a closed set. ■

THEOREM. 1.24 *Let (X, ρ) be a metric space. Then*

1. $\bar{\emptyset} = \emptyset$.
2. *For all* $A \subset X$, $A \subset \bar{A}$.
3. *If* $A \subset B \subset X$ *then* $\bar{A} \subset \bar{B}$.
4. *For* $A, B \subset X$, $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
5. *For all* $A \subset X$, $\bar{\bar{A}} = \bar{A}$.

PROOF. 1. Immediate, since $\emptyset' = \emptyset$.

2. Follows from the definition of closure.

3. $A \subset B \Rightarrow A' \subset B'$ and hence the result follows.

4. Since $A \subset A \cup B$, by (3) above, $\bar{A} \subset \overline{A \cup B}$. Similarly $\bar{B} \subset \overline{A \cup B}$, hence $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

Also if $x \in X$ such that $x \notin \bar{A} \cup \bar{B}$ then there exist $r_1, r_2 > 0$ such that $S_{r_1}(x) \cap A = \emptyset$ and $S_{r_2}(x) \cap B = \emptyset$. Let $r = \min\{r_1, r_2\}$. Then $(S_r(x) \cap A) \cup (S_r(x) \cap B) = \emptyset$, i.e., $S_r(x) \cap (A \cup B) = \emptyset$. So, $x \notin \overline{A \cup B}$ and hence $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

Thus $\bar{A} \cup \bar{B} = \overline{A \cup B}$.

5. It follows from (2) above that $\bar{A} \subset \bar{\bar{A}}$. Note that $\bar{\bar{A}} = \overline{A \cup A'} = \bar{A} \cup \bar{A'} = \bar{A} \cup (A' \cup (A')') = A \cup A' \cup (A')' = \bar{A} \cup (A')'$. So to prove the reverse inequality, it is sufficient to show that $(A')' \subset \bar{A}$. Let $x \in (A')'$. Then for any $r > 0$, $S_r(x) \cap A' \neq \emptyset$. Choose $y \in S_r(x) \cap A'$. Then by Theorem 1.5 (1) there exists $\delta > 0$ such that $S_\delta(y) \subset S_r(x)$. $S_\delta(y)$ being a neighbourhood of y, since $y \in A'$, $S_\delta(y)$ contains infinitely many elements of A (Theorem 1.18) and hence $S_r(x)$ contains infinitely many elements of A. Thus $x \in A'$, i.e., $(A')' \subset A'$. This completes the proof. ■

Theorem. 1.25 *Let $A \subset X$, where (X, ρ) is a metric space. Then*

$$\bar{A} = \cap\{F \subset X : F \textit{ is closed set}, A \subset F\}.$$

Proof. Let $x \notin \bar{A}$. Then there exists $r > 0$ such that $S_r(x) \cap A = \emptyset$, i.e., $A \subset X - S_r(x)$. Put $F = X - S_r(x)$, then F is a closed set containing A such that $x \notin F$. Thus $x \notin \cap\{F \subset X : F \text{ is closed set}, A \subset F\}$. Hence

$$\bar{A} \supset \cap\{F \subset X : F \text{ is closed set}, A \subset F\}$$

Conversely, let $x \notin \cap\{F \subset X : F \text{ is closed set}, F \supset A\}$. Then there exists a closed set $F \subset X$ such that $F \supset A$ and $x \notin F$. Then $x \in X - F$. Since $X - F$ is an open set it a neighbourhood of x, also $(X - F) \cap A = \emptyset$. Thus $x \notin \bar{A}$. Hence

$$\bar{A} \subset \cap\{F \subset X : F \text{ is closed set}, A \subset F\}.$$

Hence the result. ■

Remark. 1.26 From the above result one can conclude that $\bar{A}$ is the smallest (with respect to set inclusion) closed set containing the A.

1.1.1 Exercise

1. Show that interior of a finite set of $\mathbb{R}^n$ is empty set.

2. Show that the closure of a finite set is itself.

3. Let (X, ρ) be a metric space. Define $d : X \times X \to \mathbb{R}$ by $d(x, y) = \frac{\rho(x,y)}{1+\rho(x,y)}$ for all $x, y \in X$. Show that d is a metric on X. Show that a set V is open in (X, ρ) if and only if it is open in (X, d).

1.1.2 Subspaces of a metric space

A subset $A \subset X$, where (X, ρ) is a metric space, can be treated as a metric space whose metric is induced from the metric ρ of X.

Definition. 1.27 Let (X, ρ) be a metric space, $A \subset X$ be a non-empty subset. Then $\rho_A : A \times A \to \mathbb{R}_+$ defined by $\rho_A(a, b) = \rho(a, b)$ for all $a, b \in A$, is a metric on A, called the *induced metric on A*. The metric space (A, ρ_A) is called a *metric subspace* of the metric space (X, ρ).

Example. 1.28 1. $\mathbb{Q}$ with usual metric is a subspace of $\mathbb{R}$ with the usual metric.

2. The real line can be identified with the subset $\mathbb{R} \times \{0\} = \{(x, 0) : x \in \mathbb{R}\}$ of $\mathbb{R}^2$. The usual metric on $\mathbb{R}^2$ is the Euclidean metric d. The induced metric on $\mathbb{R} \times \{0\}$ is $d((x, y), (y, 0)) = \sqrt{(x - y)^2 + (0 - 0)^2} = |x - y|$ which is the usual metric on $\mathbb{R}$. Thus $\mathbb{R}$ is a subspace of $\mathbb{R}^2$.

Theorem. 1.29 *Let (X, ρ) be a metric space, (Y, ρ_Y) be a subspace of it. Then*

1. *For $y \in Y$, and $r > 0$, $B_{\rho_Y}(y, r) = B_\rho(y, r) \cap Y$.*
2. *A subset $V \subset Y$ is open in the subspace (Y, ρ_Y) if and only if there is an open set W in the metric space (X, ρ) such that $V = W \cap Y$.*

Proof. 1. $x \in B_{\rho_Y}(y, r) \iff x \in Y$ and $\rho_Y(x, y) < r \iff x \in Y$ and $\rho(x, y) < r \iff x \in Y$ and $x \in B_\rho(y, r) \iff x \in B_\rho(y, r) \cap Y$. Hence $B_{\rho_Y}(y, r) = B_\rho(y, r) \cap Y$.

2. Assume $V \subset Y$ is open in (Y, ρ_Y). Choose $x \in V$. Then x is an interior point of V, so there exists $r_x > 0$ such that $B_{\rho_Y}(x, r_x) \subset V$. Hence

$$\begin{aligned} V &= \cup\{B_{\rho_Y}(x, r_x) : x \in V\} = \cup\{B_\rho(x, r_x) \cap V : x \in V\} \\ &= V \cap (\cup\{B_\rho(x, r_x) : x \in V\}) \text{ (by distributive law).} \end{aligned}$$

Putting $W = \cup\{B_\rho(x, r_x) : x \in V\}$, we have W is an open set in (X, ρ) and $V = Y \cap W$.

Conversely, assume that $V = W \cap Y$ for some open set W in (X, ρ). To check that V is an open set in (Y, ρ_Y) take $x \in V$. Then $x \in W$ and hence x is an interior point of W. So there exists $r > 0$ such that $B_\rho(x, r) \subset W$. Hence $B_\rho(x, r) \cap Y \subset W \cap Y$ which implies that $B_{\rho_Y}(x, r) \subset V$. Hence x is an interior point of V with respect to ρ_Y. Since x has been chosen arbitrarily, V is open in (Y, ρ_Y). ■

Remark. 1.30 A set open in a subspace need not be open in the original metric space. For example $\mathbb{Q}$ is a subspace of $\mathbb{R}$. Though as a subset of $\mathbb{R}$, $\mathbb{Q}$ is not an open set but as a space $\mathbb{Q}$ is open. Similarly, consider the subspace $[a, b]$ of $\mathbb{R}$ and consider the open interval (c, d) where $a < c < b < d$. Then (c, d) is an open set in $\mathbb{R}$ and hence $(c, d) \cap [a, b] = (c, b]$ is open in the subspace $[a, b]$. But $(c, b]$ is never an open set in $\mathbb{R}$.

Theorem. 1.31 *Let (Y, ρ_Y) be a metric subspace of the metric space (X, ρ). Then a set $F \subset Y$ is closed in the subspace Y if and only if there exists a closed set $C \subset X$ such that $F = C \cap Y$.*

Proof. If $F = C \cap Y$ for some closed set $C \subset Y$ then F is closed in (T, ρ_Y).

Conversely, assume that F is a closed set in (Y, ρ_Y). Let $C = \mathrm{cl}_X(F)$. Then C is a closed subset of X. Since $F \subset C$ and $F \subset Y$ it follows that $F \subset C \cap Y$. Conversely, choose $x \in C \cap Y$. Then $x \in F \cup F'$ where F' denotes the derived set of F in (X, ρ). If $x \in F'$ then for any $\epsilon > 0$, $(B_\epsilon(X, \rho) - \{x\}) \cap F \neq \emptyset$ and hence $(B_\epsilon(X, \rho) \cap Y - \{x\}) \cap F \neq \emptyset$ (since $x \in Y$). This implies that $(B_\epsilon(X, \rho_Y) - \{x\}) \cap F \neq \emptyset$. Thus x is a limit point of F in (Y, ρ_Y). Since F is closed in (Y, ρ_Y) it follows that $x \in F$. Thus $C \cap Y \subset F$. Hence $F = C \cap Y$, where C is a closed set in (X, ρ). ■

The following result immediately follows from the above result.

Corollary. 1.32 *For a subset $F \subset Y$, $\mathrm{cl}_Y(F) = \mathrm{cl}_X(F) \cap Y$.*

Definition. 1.33 Let (X, ρ) be a metric space, a set $Y \subset X$ is called a *dense subset* of X if $\mathrm{cl}(Y) = X$. In this case the subspace (Y, ρ_Y) is called a *dense subspace*.

Definition. 1.34 A metric space (X, ρ) is called a *separable space* if it has a countable dense subset.

Example. 1.35 $\mathbb{R}$ with its usual metric is a separable space as $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.

Example. 1.36

1. For every $n \in \mathbb{N}$ the metric space $\mathbb{R}^n$ with Euclidean metric ρ is a separable space, $\mathbb{Q}^n$ is a dense subspace of $\mathbb{R}^n$.
2. In the metric space $C([0, 1])$ of the real valued continuous functions defined on the closed interval $[0, 1]$, the set of all polynomial functions on $[0, 1]$ is a dense subset.

1.2 Sequences, their convergence and Completeness of Metric spaces

Definition. 1.37 A *sequence* in a set X is a function $f : \mathbb{N} \to X$. If $f(n) = x_n$ for all n in $\mathbb{N}$, then one usually denote this sequence by $\{x_n\}$.

DEFINITION. 1.38 Let $\{x_n\}$ be a sequence in a metric space (X, ρ), $x_0 \in X$. Then x_0 is said to be a *cluster point* of $\{x_n\}$ if for all $\epsilon > 0$, for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m \geq n$ and $x_m \in S_\epsilon(x_0)$.

The sequence $\{x_n\}$ is said to converge to x_0 (or to be convergent with limit x_0) if for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in S_\epsilon(x_0)$ for all $n \geq n_0$. In this case x_0 is said to be the *limit* of the sequence $\{x_n\}$ and is written as $\lim_{n\to\infty} x_n = x_0$ or $\lim x_n = x_0$ or simply as $x_n \to x_0$.

It follows from the definition that if a sequence $\{x_n\}$ is convergent with x_0 as its limit then x_0 is also a cluster point of it. However a sequence having a cluster point need not be convergent.

EXAMPLE. 1.39 In $\mathbb{R}$ the sequence $\{x_n\}$, defined by $x_n = (-1)^n, n \in \mathbb{N}$, has two cluster points 1 and -1, ut it is not convergent.

THEOREM. 1.40 *The limit of a sequence is unique.*

PROOF. If possible let $\{x_n\}$ be a convergent sequence having two limits, say l and m, $l \neq m$. Choose $\epsilon = \frac{1}{2}\rho(l, m)$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that $x_n \in S_\epsilon(l)$ for all $n \geq N_1$ and $x_n \in S_\epsilon(m)$ for all $n \geq N_2$. Now, if $n > \max\{N_1, N_2\}$ then $x_n \in S_\epsilon(l) \cap S_\epsilon(m)$. This is a contradiction since $S_\epsilon(l) \cap S_\epsilon(m) = \emptyset$. ■

The next theorem follows immediately from definition of convergence.

THEOREM. 1.41 *Let (X, ρ) be a metric space, $\{x_n\}$ be a sequence in X and $l \in X$. Then the following statements are equivalent:*

1. *$\{x_n\}$ converges to l.*
2. *For every neighbourhood $\mathcal{N}_l$ of l there exists $N \in \mathbb{N}$ such that for all $n \geq N, n \in \mathbb{N}$, $x_n \in \mathcal{N}_l$.*
3. *For every open set $V \subset X$ containing l there exists $N \in \mathbb{N}$ such that for all $n \geq N, n \in \mathbb{N}$, $x_n \in V$.*

PROOF. $1 \Rightarrow 2$: Assume 1 holds and N_l is a neighbourhood of l. Then there exists $\epsilon > 0$ such that $S_\epsilon(l) \subset N_l$. Since $x_n \to l$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $n \in \mathbb{N}$, $x_n \in S_\epsilon(l)$. Hence for all $n \geq N$, $n \in \mathbb{N}$, $x_n \in N_l$.

$2 \Rightarrow 3$: Assume 2 holds. Then 3 holds immediately since any open set is a neighbourhood of each of its points.

$3 \Rightarrow 1$: Assume 3 holds. Then 1 holds immediately since each open ball is an open set. ■

REMARK. 1.42 In view of the above theorem we observe that if we know the only the neighbourhood system or only the open sets of the metric space (X,ρ) we can check whether a sequence in it is convergent or not. This helps us to define convergence of a sequence in terms of open sets or in terms of neighbourhood system.

PROBLEM. 1.43

1. Let $\{x_n\}$ be a sequence in a metric space (X,ρ) and $a \in X$. Prove that $\{x_n\}$ converges to a if and only if the sequence $\{d_n\}$ of real numbers converges to 0, where $d_n = \rho(x_n, a), n \in \mathbb{N}$.

2. Let $\{x_n\}$ be a sequence in $\mathbb{R}^2$ defined by $x_n = (\frac{n}{2n+1}, \frac{2n^2}{n^2+1})$, $n \in \mathbb{N}$. Show that the sequence $\{x_n\}$ converges to $(\frac{1}{2}, 2)$.

3. Let $\{f_n\}$ be a sequence in $B[1,2]$ where for all $n \in \mathbb{N}$, $f_n : [1,2] \to \mathbb{R}$ is defined by $f_n(x) = (1+x^n)^{1/n}$. Let $f(x) = x, \forall x \in [1,2]$. Show that $\lim\limits_{n\to\infty} f_n = f$.

Solution: Problem 1 and 2 left as an exercise.

3. Note that for all $n \in \mathbb{N}$, for all $x \in [1,2]$,

$$|f_n(x)| = |(1+x^n)^{\frac{1}{n}}| = \left| x\left(1+\frac{1}{x^n}\right)^{\frac{1}{n}} \right| \leq 2 \cdot 2^{\frac{1}{n}} \leq 4.$$

Thus $f_n \in B[1,2]$ for all $n \in \mathbb{N}$. Also it is clear that $f \in B[1,2]$. Also for all $n \in \mathbb{N}$, for all $x \in [1,2]$,

$$|f_n(x) - f(x)| = |(1+x^n)^{\frac{1}{n}} - x| = |x|\,|(1+\frac{1}{x^n})^{\frac{1}{n}} - 1| \leq 2|2^{\frac{1}{n}} - 1|.$$

So, $\sup\limits_{x\in[1,2]} |f_n(x) - f(x)| \leq 2|2^{\frac{1}{n}} - 1|\ \forall n \in \mathbb{N}$, i.e., $\rho(f_n, f) \leq 2|2^{\frac{1}{n}} - 1|$ for all $n \in \mathbb{N}$.

Hence $\lim_{n\to\infty} \rho(f_n, f) = 0$, i.e., $\lim_{n\to\infty} f_n = f$

THEOREM. 1.44 *Let A be a subset of (X,ρ), $a \in X$. Then*

1. *$a \in A'$ if and only if there exists a sequence $\{x_n\}$ of distinct elements of A which converges to a.*

2. *$a \in \bar{A}$ if and only if there exists a sequence $\{x_n\}$ in A which converges to a.*

PROOF. 1. Assume that there exists a sequence of distinct elements of A converging to A. Then for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in S_\epsilon(a)$ for all $n > N$. Hence for any $\epsilon > 0$, $S_\epsilon(a)$ contains infinitely many elements of A. Thus $a \in A'$

Conversely, let a belong to A'. Then for any $\epsilon > 0$, $S_\epsilon(a)$ contains infinitely many elements of A. Taking $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ we can find inductively $x_1 \in S_1(a)$, $x_2 \in S_{\frac{1}{2}}(a) - \{x_1\}, \ldots, x_n \in$

$S_{\frac{1}{n}}(a) - \{x_1, x_2, \ldots x_{n-1}\}, \ldots$. Thus we get a sequence $\{x_n\}$ in A such that $x_m \neq x_n$ whenever $m \neq n$ and $\rho(x_n, a) < \frac{1}{n}$. Consequently $\{x_n\}$ converges to 0.

2. Let $a \in \bar{A}$. Then for all $n \in \mathbb{N}$ choose $x_n \in S_{\frac{1}{n}}(a) \cap A$. Thus we have chosen a sequence $\{x_n\}$ in A such that $\rho(a, x_n) < \frac{1}{n}$, for all $n \in \mathbb{N}$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. So, for all $n \geq \mathbb{N}$, $\rho(a, x_n) < \frac{1}{n} < \frac{1}{N} < \epsilon$, i.e., $x_n \in S_\epsilon(a)\ \forall n \geq N$. Hence $x_n \to a$.

Conversely, let there exist a sequence in A converging to a. Then for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in S_\epsilon(a)$ for all $n > N$. Hence for any $\epsilon > 0$, $S_\epsilon(a) \cap A \neq \emptyset$. Thus $a \in \bar{A}$ ■

DEFINITION. 1.45 A sequence $\{x_n\}$ in a metric space (X, ρ) is said to be a *Cauchy sequence* if for $\epsilon > 0$ there exists a positive integer N such that $\rho(x_m, x_n) < \epsilon$ for all $m, n > N$.

THEOREM. 1.46 *Every convergent sequence in a metric space is Cauchy sequence.*

PROOF. Let $\{x_n\}$ be a convergent sequence with limit l in a metric space (X, ρ). Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\rho(x_n, l) < \frac{\epsilon}{2}$. Thus, whenever $m, n \geq N$, $\rho(x_m, x_n) \leq \rho(x_m, l) + \rho(l, x_n) < \epsilon/2 + \epsilon/2 = \epsilon$. Hence $\{x_n\}$ is a Cauchy sequence.

REMARK. 1.47 The converse of the above theorem is not true, i.e., there are metric spaces having non-convergent Cauchy sequences.

EXAMPLE. 1.48

1. Consider the set $X = \mathbb{R} - \mathbb{Q}$, of irrational numbers with usual metric on it. Let $x_n = 1 + \frac{\sqrt{2}}{n}, \forall n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence in X but having no limit in X.

2. Let for $a, b \in \mathbb{R}$, $P([a, b])$ denote the metric space of all the polynomials defined on $[a, b]$ with supnorm metric. Define $p_n(x) = (1 + \frac{x}{n})^n, x \in [a, b], n \in \mathbb{N}$. Then it can be verified that the sequence $\{p_n\}$ is a Cauchy sequence in $P([a, b])$ and $\lim p_n$ does not exist in $P([a, b])$. (In fact $\lim p_n(x) = e^x, x \in [a, b]$ and $e^x \notin P([a, b])$).

DEFINITION. 1.49 A metric space (X, ρ) is said to be *complete* if each Cauchy sequence in X converges to a point of X.

A metric space which is not complete is called an incomplete metric space.

EXAMPLE. 1.50 The set of real numbers, the set of complex numbers with usual metric are examples of complete metric spaces, whereas the set of rational numbers with usual metric is an example of incomplete metric space.

THEOREM. 1.51 *Let (X, ρ) be a complete metric space and $A \subset X$. Then (A, ρ_A) is a complete metric space if and only if A is closed in (X, ρ).*

Proof. Suppose (A, ρ_A) is complete, $\xi \in \bar{A}$. Then there exists a sequence $\{x_n\}$ in A such that $\lim x_n = \xi$. Clearly $\{x_n\}$ is a Cauchy sequence in A. By completeness of (A, ρ_A) there exists a in A such that $\lim x_n = a$. Hence $\xi = a$, i.e., $\xi \in A$. So $\bar{A} \subset A$, i.e., A is closed.

Conversely, let A be a closed subset of (X, ρ), $\{x_n\}$ be a Cauchy sequence in (A, ρ_A). Then $\{x_n\}$ is a Cauchy sequence in (X, ρ). Since (X, ρ) is complete there exists $x_0 \in X$ such that $\lim x_n = x_0$. As $x_n \in A$ for all n in $\mathbb{N}$, $x_0 \in \bar{A}$. Since A is closed $\bar{A} = A$ and hence $x_0 \in A$. Thus the Cauchy sequence $\{x_n\}$ in (A, ρ_A) converges to a point of A. Hence (A, ρ_A) is complete. ■

Definition. 1.52 Let A be a non-empty subset of a metric space (X, ρ). Then A is said to have a *finite diameter* if $\{\rho(x, y) : x, y \in A\}$ is bounded. Otherwise A is said to have *infinite diameter*. If A has finite diameter, then $\sup\{\rho(x, y) : x, y \in A\}$ is called the *diameter* of A and is denoted by $\rho(A)$. By definition we shall assume that $\rho(\emptyset) = -\infty$.

Problem. 1.53 Show that for any $A \subset X$, where (X, ρ) is a metric space, $\rho(A) = \rho(\bar{A})$.

Solution: As $A \subset \bar{A}$, it immediately follows that $\rho(A) \leq \rho(\bar{A})$.

If possible let $\rho(A) \lneqq \rho(\bar{A})$. Choose $\epsilon > 0$ such that $\rho(A) + \epsilon < \rho(\bar{A})$. Then one can choose $x, y \in \bar{A}$ such that $\rho(A) + \epsilon < \rho(x, y)$. Now, one can choose $a, b \in A$ such that $\rho(a, x) < \epsilon/2$ and $\rho(b, y) < \epsilon/2$. Also $\rho(a, b) \leq \rho(A)$. Hence,

$$\begin{aligned} \rho(x, y) &\geq \rho(A) + \epsilon > \rho(a, b) + \epsilon/2 + \epsilon/2 > \rho(a, b) + \rho(b, y) + \rho(a, x) \\ &\geq \rho(x, y) \quad \text{— a contradiction.} \end{aligned}$$

Thus $\rho(A) \not< \rho(\bar{A})$, i.e., $\rho(A) \geq \rho(\bar{A})$.

Problem. 1.54 If $\{x_n\}$ is a sequence in a metric space (X, ρ) and $p \in X$ such that $S_{\frac{1}{n}}(p) \cap S_{\frac{1}{n}}(x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, prove that $\lim x_n = p$.

Theorem. 1.55 *If a Cauchy sequence has a cluster point then the sequence converges to it.*

Proof. Let $\{x_n\}$ be a Cauchy sequence having a cluster point p. Let $\epsilon > 0$ be given. Then there exists $k \in \mathbb{N}$ such that $\rho(x_m, x_n) < \epsilon/2$ for all $m, n > k$. Since p is a cluster point, there exists $s > k$ such that $\rho(x_s, p) < \epsilon/2$. Thus, for all $n > k$, $\rho(x_n, k) \leq \rho(x_n, x_s) + \rho(x_s, p) < \epsilon/2 + \epsilon/2 = \epsilon$. Thus $\lim x_n = p$. ■

Remark. 1.56 A Cauchy sequence can have at most one cluster point.

Theorem. 1.57 [Cantor Intersection Property] *Let (X, ρ) be a metric space. Then the followings are equivalent:*

1. *(X, ρ) is complete.*

2. *If $\{F_n\}$ is a sequence of non-empty closed sets in (X, ρ) such that $F_1 \supset F_2 \supset F_3 \supset \cdots$ and $\rho(F_n) \to 0$ as $n \to \infty$, then $\cap_{n=1}^{\infty} F_n$ constitutes of exactly one point.*

Proof. 1. $\Rightarrow$ 2.: Suppose that (X, ρ) is complete.

If possible, let $F_1 \cap F_2 \cap F_3 \cap \cdots$ contains two distinct points, say, x and y. Then $\rho(F_n) \geq \rho(x, y)$ for all $n \in \mathbb{N}$, which shows that $\lim \rho(F_n) \geq \rho(x, y) > 0$ – a contradiction. Thus the intersection $F_1 \cap F_2 \cap F_3 \cdots$ can contain at most one point.

For each $n \in \mathbb{N}$ choose $x_n \in F_n$. (Such a choice is possible by axiom of choice). Let $\epsilon > 0$ be given. Then there exists $k \in \mathbb{N}$ such that $\rho(F_k) < \epsilon$. If $m, n > k$ then $x_m \in F_m \subset F_k$ and $x_n \in F_n \subset F_k$, which shows that $\rho(x_m, x_n) \leq \rho(F_k) < \epsilon$, i.e., $\{x_n\}$ is a Cauchy sequence. Since (X, ρ) is complete, there exists $l \in X$ such that $\lim x_n = l$. Let $p \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, $x_{n+p} \in F_p$. Also $\lim x_n = \lim x_{n+p} = l$ Hence $l \in \bar{F}_p = F_p$. Since p has been chosen arbitrarily, $l \in F_n$ for all $n \in \mathbb{N}$, thus $p \in \cap\{F_n : n \in \mathbb{N}\}$.

2. $\Rightarrow$ 1.: Conversely, suppose that condition 2 holds.

Let $\{x_n\}$ be a Cauchy sequence in (X, ρ). Put $A_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$, for all $n = 1, 2, 3, \ldots$. Then $\bar{A}_1 \supset \bar{A}_2 \supset \bar{A}_3 \supset \cdots$. Let $\epsilon > 0$ be a real number. The there exists $k \in \mathbb{N}$ such that $\rho(x_m, x_n) < \epsilon/2$ for all $m, n \geq k$ and hence $\rho(\bar{A}_n) = \rho(A_n) \leq \epsilon/2 < \epsilon$ for all $n > k$. Thus $\rho(\bar{A}_n) \to 0$ as $n \to \infty$. By condition 2 there exists $x_0 \in \cap\{\bar{A}_n : n \in \mathbb{N}\}$. So x_0 is a cluster point of the sequence $\{x_n\}$. Since $\{x_n\}$ is a Cauchy sequence, $x_n \to x_0$. Thus $X, \rho)$ is complete. ■

Definition. 1.58 Let (X, ρ) is a metric space, $A \subset X$. Then A is said to be *dense* in (X, ρ) if $\bar{A} = X$. A is said to be *nowhere dense* if $(\bar{A})^\circ = \emptyset$.

Problem. 1.59 Let (X, ρ) be a metric space, $A \subset X$.

1. Show that A is dense in (X, ρ) if and only if $V \cap A \neq \emptyset$ for all open set $V \subset X$.

2. For any open set $V \subset X$, show that $V \cap A \neq \emptyset$ if and only if $V \cap \bar{A} \neq \emptyset$.

3. Show that A is nowhere dense if and only if $V \not\subset A$ for any open set $V \subset X$.

4. Prove that A is nowhere dense if and only if for all open set $V \subset X$ there exists $S_r(x) \subset V$ such that $S_r(x) \cap A = \emptyset$.

DEFINITION. 1.60 Let V be a vector space over the field of real or complex numbers. A real valued function $x \mapsto \|x\|$ defined on V is called a *norm* on V if the following conditions are satisfied:

1. For all $x \in V$, $\|x\| \geq 0$ equality holds if and only if $x = 0$.
2. For any scalar λ for any $x \in V$, $\|\lambda x\| = |\lambda| \, \|x\|$.
3. For all $x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$.

A real or complex vector space with a norm $\|, \|$ defined on it is called a *normed linear space* and is usually denoted by $(V, \|, \|)$ or simply by V.

The following result can easily be verified.

THEOREM. 1.61 *Let $(V, \|, \|)$ be a normed linear space. Define $\rho(x, y) = \|x - y\|$ for all $x, y \in V$. Then ρ is a metric on V.*

DEFINITION. 1.62 The metric defined above is called *metric induced by norm.*

EXAMPLE. 1.63

1. In $\mathbb{R}^n$ we define

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}, \ \ \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,$$

   one can verify (using Schwartz's inequality) that $\|, \|$ is a norm on $\mathbb{R}^n$. This is called the usual norm on $\mathbb{R}^n$. It can easily be verified that the metric on $\mathbb{R}^n$ induced by usual norm is nothing but the usual metric on it.

2. Recall that $B[a, b]$, the set of all real valued bounded functions defined on $[a, b]$ is a linear space over the field of real numbers. For $f \in B[a, b]$ we define

$$\|f\| = \sup\{|f(x) : a \leq x \leq b\}.$$

   this norm is known as *supnorm.*

THEOREM. 1.64 *$B[a, b]$ is a complete metric space (with respect to supnorm metric).*

PROOF. Let $\{f_n\}$ be a Cauchy sequence in $B[a, b]$. Let $\epsilon > 0$ be given. Then there exists a positive integer n_1 such that $\|f_n - f_m\| < \frac{\epsilon}{3}$ for all $m, n \geq n_1$. Let $x \in [a, b]$, then $|f_n(x) - f_m)| \leq \|f_n - f_m\| < \frac{\epsilon}{3}$ for all $m, n \geq n_1$. This shows that for any $x \in [a, b]$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$.

By completeness of $\mathbb{R}$ the sequence $\{f_n(x)\}$ is convergent. Let us define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = \lim_{n \to \infty} f_n(x) \ \ \forall x \in [a, b].$$

It remains to show that $g \in B[a, b]$ and $\{f_n\} \to g$ with respect to supnorm metric. Let $x \in [a, b]$. Since $\{f_n(x)\} \to g$ there exists $n_x > n_1$ such that $|f_n(x) - g(x)| < \frac{\epsilon}{3}$ for all $n \geq n_x$, in particular, $|f_{n_x}(x) - g(x)| < \frac{\epsilon}{3}$. Thus,

$$\begin{aligned} |f_n(x) - g(x)| &\leq |f_n(x) - f_{n_x}(x)| + |f_{n_x}(x) - g(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} \quad \forall x \in [a, b] \ \ \forall n > n_1. \end{aligned}$$

Thus, $\sup\{|f_n(x) - g(x)| : a \leq x \leq b\} \leq \dfrac{2\epsilon}{3} < \epsilon \ \ \forall n > n_1$, i.e.,

$$\|f_n - g\| < \epsilon \ \forall n > n_1.$$

This shows that $\{f_n\} \to g$ with respect to supnorm metric. Also for any $x \in [a, b]$, $|g(x)| \leq \|g\| \leq \|f_{n_1} - g\| + \|f_{n_1}\| < \epsilon + \|f_{n_1}\|$, which shows that $g \in B[a, b]$.

This completes the proof. ■

Let us denote by $C[a, b]$ the set of all continuous real valued functions defined on $[a, b]$. Then $C[a, b] \subset B[a, b]$.

Theorem. 1.65 *The set $C[a, b]$ is closed in $B[a, b]$.*

Let $f \in B[a, b]$ such that $f \in \overline{C[a, b]}$. Let $\epsilon > 0$ be a real, choose $g \in C[a, b]$ such that $\|f - g\| < \frac{\epsilon}{3}$. Let $x_0 \in [a, b]$, by continuity of g there exists $\delta > 0$ such that $|g(x) - g(x_0)| < \frac{\epsilon}{3}$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$. Now for any x in $(x_0 - \delta, x_0 + \delta) \cap [a, b]$,

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - g(x)| + |g(x) - g(x_0)| + |g(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This shows that f is continuous at x_0. Since x_0 has been chosen arbitrarily in $[a, b]$, f is continuous on $[a, b]$. Hence $f \in C[a, b]$.

Thus $C[a, b]$ is closed in $B[a, b]$. ■

Theorem. 1.66 *For $a, b \in \mathbb{R}$, $C[a, b]$ is a complete metric space.*

Proof. $C[a, b]$ is a closed subspace of $B[a, b]$. $B[a, b]$ is a complete metric space. So $C[a, b]$ is a complete metric space.

Problem. 1.67

1. Write down an independent proof of the fact that $C[a, b]$ is a complete metric space with respect to supnorm metric.
2. Prove that $\mathbb{R}^n$ is a complete metric space for all $n = 1, 2, 3 \ldots$

2 Continuity, Connectedness, Compactness and Fixed Point Theorem

2.1 Continuity

Before going to the definition of continuity we introduce some notations and establish some results.

Let $f : X \to Y$ be a function, $A \subset X, B \subset Y$ then

$$f(A) = \{f(x) : x \in A\} \quad \text{and} \quad f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

If $A_1, A_2 \subset X$, $B_1, B_2 \subset Y$, then

(a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

(c) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

(d) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

There are examples where equality in (b) does not hold.

Definition. 2.1 Let $(X, \rho), (Y, \sigma)$ be two metric spaces, $f : X \to Y$ be a function, $a \in X$. f is said to be *continuous* at a if for any $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x \in X$, $\rho(x, a) < \delta \Rightarrow \sigma(f(x), f(a)) < \epsilon$.

The function f is said to be *continuous on* X if f is continuous at each point of X.

Theorem. 2.2 *Let $f : (X, \rho) \to (Y, \sigma)$ be a function and $a \in X$. Then the following are equivalent:*

1. *f is continuous at a.*
2. *For all $\epsilon > 0$ there exists $\delta > 0$ such that $f(S_\delta(a)) \subset S_\epsilon(f(a))$.*
3. *For all neighbourhood $N_{f(a)}$ of $f(a)$, $f^{-1}(N_{f(a)})$ is a neighbourhood of a.*
4. *For any sequence $\{x_n\}$ in X, $\lim x_n = a \Rightarrow \lim f(x_n) = f(a)$.*
5. *for all $A \subset X$, $a \in \bar{A} \Rightarrow f(a) \in \overline{f(A)}$.*

Proof. $1 \Rightarrow 2$: Assume f is continuous at a, $\epsilon > 0$. then there exists $\delta > 0$ such that for all $x \in X$, $\rho(x, a) < \delta \Rightarrow \sigma(f(x), f(a)) < \epsilon$. So $x \in S_\delta(a) \Rightarrow \rho(x, a) < \delta \Rightarrow \sigma(f(x), f(a)) < \epsilon \Rightarrow f(x) \in S_\epsilon(f(a))$.

$2 \Rightarrow 3$: Assume 2 holds, $N_{f(a)}$ is a neighbourhood of $f(a)$. Then there exists $\epsilon > 0$ such that $S_\epsilon(f(a)) \subset N_{f(a)}$. By 2 there exists $\delta.0$ such that $f(S_\delta(a)) \subset S_\epsilon(f(a))$, i.e., $f(S_\delta(a)) \subset N_{f(a)}$ and hence $S_\delta(a) \subset f^{-1}(N_{f(a)})$. Thus $f^{-1}(N_{f(a)})$ is a neighbourhood of a.

$3 \Rightarrow 4$: Let 3 hold. $\{x_n\}$ be a sequence in X converging to a. Let $\epsilon > 0$. Then $S_\epsilon(f(a))$ is a neighbourhood of $f(a)$. By 3 $f^{-1}(S_\epsilon(f(a))$ is a neighbourhood of a. Since $x_n \to a$, there exists $N \in \mathbb{N}$ such that $x_n \in f^{-1}(S_\epsilon(f(a))$ for all $n \geq N$. Thus $f(x_n) \in S_\epsilon(f(a))$ for all $n \geq N$. Hence $\{f(x_n)\}$ converges to $f(a)$.

$4 \Rightarrow 5$: Let $a \in \bar{A}$. Then there exists a sequence $\{x_n\}$ in A which converges to A. By 4 the sequence $\{f(x_n)\}$ converges to $f(a)$. But $\{f(x_n)\}$ is a sequence in $f(A)$, thus $f(a)$ belongs to $\overline{f(A)}$.

$5 \Rightarrow 1$: Assume 5 holds. If possible suppose that f is not continuous at a. Then there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x \in X$ such that $\rho(x, a) < \delta$ but $\sigma(f(x), f(a)) \geq \epsilon$. In particular, for all $n \in \mathbb{N}$ there exists $x_n \in X$ such that $\rho(x_n, a) < \frac{1}{n}$ but $\sigma(f(x_n), f(a)) \geq \epsilon$. Put $A = \{x_1, x_2, \ldots\}$. Then since $\rho(x_n, a) < \frac{1}{n} \to 0$ as $n \to \infty$, $a \in \bar{A}$. Also $f(A) = \{f(x_1), f(x_2), \ldots\}$ and $S_\epsilon(f(a)) \cap f(A) = \emptyset$ which shows that $f(a) \notin \overline{f(A)}$ — a contradiction. Thus $5 \Rightarrow 1$. ■

Definition. 2.3 Let $(X, \rho), (Y, \sigma)$ be two metric spaces, $a \in X$ and $f : X \to Y$ be a function. f is said to *preserve convergence* at a if for any sequence $\{x_n\}$ converging to a the sequence $\{f(x_n)\}$ converges to $f(a)$. f is said to *preserve nearness* at a if for any $A \subset X$, $a \in \bar{A}$ implies that $f(a) \in \overline{f(A)}$.

In view of the last result one can say that a function f is continuous at a point if and only if it preserves convergence at that point if and only if it preserves nearness at that point.

Theorem. 2.4 *Let $(X, \rho), (Y, \sigma)$ be two metric spaces, $f : X \to Y$ be a function. Then the followings are equivalent:*

1. *f is continuous on X.*
2. *For all open $V \subset Y$, $f^{-1}(V)$ is open in (X, ρ).*
3. *For all closed $F \subset Y$, $f^{-1}(F)$ is closed in (X, ρ).*

Proof. $1 \Rightarrow 2$: Assume that 1 holds, $V \subset Y$ is open. Let $a \in f^{-1}(V)$. Then $f(a) \in V$. Since V is open there exists $\epsilon > 0$ such that $S_\epsilon(f(a)) \subset V$. By continuity of f at a, there exists $\delta > 0$ such that $f(S_\delta(a)) \subset S_\epsilon(f(a)) \subset V$ which implies that $S_\delta(a) \subset f^{-1}(V)$. Thus a is an interior point of $f^{-1}(V)$. Since a has been chosen arbitrarily in $f^{-1}(V)$, each point of it is interior point of it. Thus $f^{-1}(V)$ is open.

$2 \Rightarrow 3$: Assume that 2 holds, $F \subset Y$ is closed. Then $V = Y - F$ is an open set. By 2 $f^{-1}(V)$ is an open set in (X, ρ). But $f^{-1}(V) = f^{-1}(Y - F) = X - f^{-1}(F)$. Thus $f^{-1}(F)$ is a closed set.

$3 \Rightarrow 1$: Assume that 3 holds, $a \in X$. If possible, suppose that f be not continuous at a. Then there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x \in X$ such that $\rho(a, x) < \delta$ but $\sigma(f(a), f(x)) \geq \epsilon$. In particular, taking $\delta = \frac{1}{n}$, we get for all n in $\mathbb{N}$, $x_n \in X$ such taht $\rho(a, x_n) < \frac{1}{n}$ but $\sigma(f(a), f(x_n)) \geq \epsilon$. Take $A = \{f(x_n) : n \in \mathbb{N}\}$ and $F = \bar{A}$. Then F is a closed set in (Y, σ) and $f(a) \notin F$. Note that $x_n \in f^{-1}(F)$ for all $n \in \mathbb{N}$. Since $x_n \to a$, $a \in \overline{f^{-1}(F)}$ but $a \notin f^{-1}(F)$ which shows that $f^{-1}(F)$ is not a closed set — a contradiction. Thus 1 holds. ■

The next theorem states that the composition of two continuous functions is again a continuous function.

THEOREM. 2.5 *Let $f : (X, \rho) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu)$ be continuous functions. Then $g \circ f : (X, \rho) \to (Z, \mu)$ is continuous.*

PROOF. Let V be an open set in (Z, μ). Then by continuity of g, $g^{-1}(V)$ is an open set in (Y, σ). Again by continuity of f, $f^{-1}(g^{-1}(V))$ is open in (X, ρ), i.e., $(f^{-1} \circ g^{-1})(V)$ is open in (X, ρ). Since $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, the result follows. ■

THEOREM. 2.6 *Let $f : (X, \rho) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu)$ be two functions, $a \in X$. If f is continuous at a and g is continuous at $f(a)$ then $g \circ f$ is continuous is continuous at a.*

PROOF. Let $\{x_n\}$ be a sequence in X converging to a. Then by continuity of f at a the sequence $\{f(x_n)\}$ in Y converges to $f(a)$. Again by continuity of g at the point $f(a)$ the sequence $\{g(f(x_n))\}$ in Z converges to $g(f(a))$. Thus for every sequence $\{x_n\}$ in X converging to a the sequence $\{(g \circ f)(x_n)\}$ in Z converges to $(g \circ f)(a)$. So $g \circ f$ is continuous at a. ■

DEFINITION. 2.7 Let $f : (X, \rho) \to (Y, \sigma)$ be a function. Then f is said to be *uniformly continuous* on X if for all $\epsilon > 0$ there exists $\delta > 0$ (depending on ϵ only) such that

$$\forall x_1, x_2 \in X, \rho(x_1, x_2) < \delta \Rightarrow \sigma(f(x_1), f(x_2)) < \epsilon.$$

It can be noted that every uniformly continuous function is continuous. the converse is not true, i.e., there are functions which are continuous but not uniformly continuous. An example of such a function will be given later.

THEOREM. 2.8 *If $f : (X, \rho) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu)$ are two uniformly continuous functions then $g \circ f : (X, \rho) \to (Z, \mu)$ is uniformly continuous.*

PROOF. Let $\epsilon > 0$ be a real number. By uniform continuity of g there exists $\delta_1 > 0$ such that for all $y_1, y_2 \in Y$, $\sigma(y_1, y_2) < \delta_1 \Rightarrow \mu(g(y_1), g(y_2)) < \epsilon$. Again by unifor continuity of f there exists $\delta > 0$ such that for all $x_1, x_2 \in X$, $\rho(x_1, x_2) < \delta \Rightarrow \sigma(f(x_1), f(x_2)) < \delta_1$. Thus for all $x_1, x_2 \in X$, $\rho(x_1, x_2) < \delta \Rightarrow \sigma(f(x_1), f(x_2)) < \delta_1 \Rightarrow \mu(g(f(x_1)), g(f(x_2))) < \epsilon$, i.e., $\rho(x_1, x_2) < \delta \Rightarrow \mu(g \circ f(x_1)), g \circ f(x_2))) < \epsilon$. Hence $g \circ f$ is uniformly continuous. ■

Before going to next result we define the distance between two sets.

DEFINITION. 2.9 Let Let A, B be two nonempty subsets of a metric space (X, ρ). Then the *distance between two sets* A and B is denoted by $\rho(A, B)$ and is defined by

$$\rho(A, B) \quad = \quad \inf\{\rho(a, b) : a \in A, b \in B\}.$$

By definition we assume that $\rho(A, \emptyset) = \infty$. For $x \in X$, we write $\rho(x, A)$ for $\rho(\{x\}, A)$.

THEOREM. 2.10 *Let (X, ρ) be a metric space, $A \subset X, A \neq \emptyset$. Then the function $f : X \to \mathbb{R}$, defined by $f(x) = \rho(x, A)\ \forall x \in X$, is uniformly continuous.*

PROOF. Let $x_1, x_2 \in X$ and $a \in A$. Then

$$\begin{aligned}
& \rho(x_1, a) \leq \rho(x_1, x_2) + \rho(x_2, a) \\
\Rightarrow\ & \inf\{\rho(x_1, a) : a \in A\} \leq \rho(x_1, x_2) + \inf\{\rho(x_2, a) : a \in A\} \\
\Rightarrow\ & \rho(x_1, A) \leq \rho(x_1, x_2) + \rho(x_2, A) \\
\Rightarrow\ & \rho(x_1, A) - \rho(x_2, A) \leq \rho(x_1, x_2) \\
\Rightarrow\ & f(x_1) - f(x_2) \leq \rho(x_1, x_2).
\end{aligned}$$

Similarly, $f(x_2) - f(x_1) \leq \rho(x_1, x_2)$ and hence $|f(x_1) - f(x_2)| \leq \rho(x_1, x_2)$ for all $x_1, x_2 \in X$. Thus f is uniformly continuous. ■

COROLLARY. 2.11 *Let A be a nonempty subset of a metric space (X, ρ). Then*

$$\bar{A} \quad = \quad \{x \in X : \rho(x, A) = 0\}.$$

PROOF. Let $x \in X$ and $\rho(x, A) = 0$. Then $S_\epsilon(x) \cap A \neq \emptyset$ for all $\epsilon > 0$, thus $x \in \bar{A}$. Hence $\{x \in X : \rho(x, A) = 0\} \subset \bar{A}$. ■

Note that $f : X \to \mathbb{R}$ defined by $f(x) = \rho(x, A)$ is continuous. Since for any $x \in A$ $f(x) = 0$, it follows that $A \subset f^{-1}(\{0\})$. Also $\{0\}$ being a closed set $f^{-1}(\{0\})$ is closed and hence $\bar{A} \subset f^{-1}(\{0\}) = \{x \in X : \rho(x, A) = 0\}$. So $\bar{A} = \{x \in X : \rho(x, A) = 0\}$.

An important property of a uniformly continuous function is that it carries Cauchy sequences to Cauchy sequences.

Theorem. 2.12 *Let $f : (X, \rho) \to (Y, \sigma)$ be a uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in (X, ρ) then $\{f(x_n)\}$ is a Cauchy sequence in (Y, σ).*

Proof. Let $\epsilon > 0$ be a real. Then there exists $\delta > 0$ such that for all $x, y \in X$, $\sigma(f(x), f(y)) < \epsilon$ whenever $\rho(x, y) < \delta$. Since $\{x_n\}$ is a cauchy sequence one can find $N \in \mathbb{N}$ such that $\rho(x_m, x_m) < \delta$ for all $m, n \geq N$. Hence $\sigma(f(x_m), f(x_n)) < \epsilon$ for all $m, n \geq N$. Thus $\{f(x_n)\}$ is a Cauchy sequence in (Y, σ). ■

Below we give an example of continuous function which is not uniformly continuous.

Example. 2.13 define $f : \mathbb{R} - \{0\} \to \mathbb{R}$ by $f(x) = \frac{1}{x}, \forall x \in \mathbb{R} - \{0\}$. Then f is continuous on $\mathbb{R} - \{0\}$. Note that $\{\frac{1}{n}\}$ is a Cauchy sequence in $\mathbb{R} - \{0\}$. Also note that $\{f(\frac{1}{n})\} = \{n\}$ which is not a Cauchy sequence in $\mathbb{R}$. Thus f is not uniformly continuous.

Definition. 2.14 Let k be a positive integer. For all $i \in \{1, 2, \ldots, k\}$ let us define a map $\pi_i : \mathbb{R}^k \to \mathbb{R}$ by $\pi_i(x) = x_i \; \forall x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$. π_i is called the i-th *projection map* from $\mathbb{R}^k$ to $\mathbb{R}$.

Theorem. 2.15 *The projection map $\pi_i : \mathbb{R}^k \to \mathbb{R}$, $1 \leq i \leq k$, is uniformly continuous.*

Proof. If $x = (x_1, x_2, \ldots, x_k), y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k$, $1 \leq i \leq k$, then $|\pi_i(x) - \pi_i(y)| = |x_i - y_i| \leq \sqrt{\sum_{j=i}^{k}(x_j - y_j)^2} = ||x - y||$. Hence the result follows. ■

Theorem. 2.16 *Let (X, ρ) be a metric space, k be a positive integer and $f : (X, \rho) \to \mathbb{R}^k$ be a function. Then f is uniformly continuous if and only if $\pi_i \circ f : X \to \mathbb{R}$ is uniformly continuous for all $i = 1, 2, \ldots, k$.*

If f is uniformly continuous then, since π_i is uniformly continuous for all $i = 1, 2, \ldots, k$, it follows that $\pi_i \circ f$ is uniformly continuous for all $i = 1, 2, \ldots, k$.

Conversely, suppose that $\pi_i \circ f$ is uniformly continuous for all $i = 1, 2, \ldots, k$. Let $\epsilon > 0$ be a real. Then for each $i \in \{1, 2, \ldots, k\}$ there exists $\delta_i > 0$ such that for all $x, y \in X$, $\rho(x, y) < \delta_i \Rightarrow |\pi_i \circ f(x) - \pi_i \circ f(y)| < \frac{\epsilon}{\sqrt{k}}$. Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_k\}$. Then for $x, y \in X$,

$$\rho(x, y) < \delta \quad \Rightarrow \quad |\pi_i \circ f(x) - \pi_i \circ f(y)| < \frac{\epsilon}{\sqrt{k}} \qquad \forall i = 1, 2, \ldots, k.$$

Hence if $\rho(x, y) < \delta$ then

$$||f(x) - f(y)|| \;=\; \sqrt{\sum_{i=1}^{k}(\pi_i \circ f(x) - \pi_i \circ f(y))^2} \;<\; \epsilon.$$

This shows that f is uniformly continuous. ■

2.2 Connectedness

DEFINITION. 2.17 A metric space (X, ρ) is said to be *connected* if there exist no open sets $G_1, G_2 \subset X$ such that $G_1 \neq \emptyset \neq G_2$, $G_1 \cap G_2 = \emptyset$ and $X = G_1 \cup G_2$, i.e., if X can not be expressed as a union of two disjoint non-empty open sets.

A metric space or its subset which is not connected is called *disconnected*.

It can be noted that if X is disconnected metric space then X is expressed as $X = G_1 \cup G_2$, where G_1, G_2 are disjoint non-empty open sets. Thus $G_1 = X - G_2$, since G_2 is an open set its complement G_1 is a closed set. Thus G_1 is an open set as well as a closed set, called a *clopen set*. Similarly G_2 is also a clopen set.

On the other hand if X contains a clopen set V, $\emptyset \neq V \neq X$, Then V and $X - V$ are both non-empty open sets and $X = V \cup (X - V)$ which shows that X is a disconnected metric space.

Thus we conclude that

THEOREM. 2.18 *A metric space is disconnected if and only if it contains a non-trivial clopen set.*

DEFINITION. 2.19 Let (X, ρ) be a metric space. A set $A \subset X$ is said to be *connected* if the subspace (A, ρ_A) is connected.

THEOREM. 2.20 *Let A be a subset of a metric space (X, ρ). Then A is connected if and only if there exist no two open sets $G_1, G_2 \subset X$ such that $A \subset G_1 \cup G_2$, $A \cap G_1 \neq \emptyset \neq A \cap G_2$ and $A \cap G_1 \cap G_2 = \emptyset$.*

PROOF. Suppose that A is connected, i.e., (A, ρ_A) is a connected metric space. If possible, let there exist open sets G_1, G_2 in (X, ρ) such that $A \subset G_1 \cup G_2$, $A \cap G_1 \neq \emptyset \neq A \cap G_2$ and $A \cap G_1 \cap G_2 = \emptyset$. Put $V_1 = A \cap G_1, V_2 = A \cap G_2$. Then V_1, V_2 are open sets in (A, ρ_A), $V_1 \neq \emptyset \neq V_2$, $V_1 \cap V_2 = \emptyset$ and $A = V_1 \cup V_2$. Thus (A, ρ_A) is not connected — a contradiction.

Conversely suppose that the condition of the theorem holds. We claim that A is connected. If not, then (A, ρ_A) is not connected and hence there exists non-empty open sets V_1, V_2 in (A, ρ_A) such that $A = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. So there exists open sets G_1, G_2 in (X, ρ) such that $V_1 = A \cap G_1, V_2 = A \cap G_2$. Thus we found two open sets $G_1, G_2 \subset X$ such that $A \subset G_1 \cup G_2$, $A \cap G_1 \neq \emptyset \neq A \cap G_2$ and $A \cap G_1 \cap G_2 = \emptyset$ — a contradiction. ■

COROLLARY. 2.21 *If A is an open set in a metric space (X, ρ), then A is connected if and only if there exist no two open sets $G_1, G_2 \subset A$ such that $A = G_1 \cup G_2$, $G_1 \neq \emptyset \neq G_2$ and $G_1 \cap G_2 = \emptyset$.* ■

REMARK. 2.22 If (X, ρ) is a metric space and $a \in X$ then $\{a\}$ is connected. Also the empty subset of each metric space is connected.

EXAMPLE. 2.23 Let A be a set of rationals containing more than one point. Then A is not connected in $\mathbb{R}$.

Let $x, y \in A$ such that $x < y$. Choose an irrational ξ such that $x < \xi < y$. Set $G_1 = (-\infty, \xi), G_2 = (\xi, \infty)$. Then $A \subset G_1 \cup G_2$, $A \cap G_1 \neq \emptyset \neq A \cap G_2$ and $A \cap G_1 \cap G_2 = \emptyset$. So A is not connected.

THEOREM. 2.24 *Let I be a subset of $\mathbb{R}$. Then I is connected if and only if I is an interval in $\mathbb{R}$.*

PROOF. Recall that a set $I \subset \mathbb{R}$ is called an interval if for all $x, y \in I$ for all $z \in \mathbb{R}$, $x < z < y$ implies that $z \in I$.

Let I be a connected subset of $\mathbb{R}$. We claim that I is an interval of $\mathbb{R}$. If not, then there exists $a, b \in I$, $x \in \mathbb{R}$ such that $a < x < b$ and $x \notin I$. Set $G_1 = (-\infty, x), G_2 = (x, \infty)$. The G_1, G_2 are open sets in $\mathbb{R}$ such that $I \subset G_1 \cup G_2$, $I \cap G_1 \neq \emptyset \neq I \cap G_2$ and $I \cap G_1 \cap G_2 = \emptyset$. Thus I is not connected — a contradiction.

Conversely, let I be an interval of $\mathbb{R}$. If $I = \emptyset$ or I contains a single point then I is connected. Let I contains more than one point. If possible, suppose that I is not connected. Then there exist open sets $G_1, G_2 \subset \mathbb{R}$ such that $I \subset G_1 \cup G_2$, $I \cap G_1 \neq \emptyset \neq I \cap G_2$ and $I \cap G_1 \cap G_2 = \emptyset$. Choose $a \in I \cap G_1, b \in I \cap G_2$. Since $a \neq b$, without any loss of generality we may assume that $a < b$. Let $c = \frac{a+b}{2}$. Then $a < c < b$. I being an interval, $c \in I$. So, either $c \in G_1$ or $c \in G_2$, also $c \notin G_1 \cap G_2$. If $c \in G_1$, we put $a_1 = c, b_1 = b$, if $c \in G_2$, put $a_1 = a, b_1 = c$. In any case we found reals $a_1 \in G_1, b_1 \in G_2$, $[a_1, b_1] \subset [a, b]$ and $b_1 - a_1 = \frac{1}{2}(b - a)$.

Suppose that we have found reals $a_1, a_2, \ldots, a_n \in G_1$, $b_1, b_2, \ldots, b_n \in G_2$ such that $[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n]$ and $b_k - a_k = \frac{b-a}{2^k}$ for $k = 1, 2, \ldots, n$. Put $c_n = \frac{1}{2}(a_n + b_n)$. then $c_n \in I \subset G_1 \cup G_2$. Also $c_n \notin G_1 \cap G_2$. If $c_n \in G_1$, put $a_{n+1} = c_n, b_{n+1} = b_n$, if $c_n \in G_2$, put $a_{n+1} = a_n, b_{n+1} = c_n$. Thus $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ and $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) = \frac{1}{2^{n+1}}(b - a)$.

Thus using induction we can conclude that there exist a sequence of closed intervals such that $a_n \in G_1, b_n \in G_2 \ \forall n \in \mathbb{N}$, $[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots$ and $b_n - a_n = \frac{b-a}{2^n} \ \forall n \in \mathbb{N}$. By Cantor's Intersection property there exists $\xi \in [a_n, b_n] \ \forall n \in \mathbb{N}$, i.e., $a_n \leq \xi \leq b_n \ \forall n \in \mathbb{N}$. Also $\lim_{n \to \infty} a_n = \xi = \lim_{n \to \infty} b_n$. So $\xi \in I \subset G_1 \cup G_2$. So either $\xi \in G_1$ or $\xi \in G_2$.

If $\xi \in G_1$ then there exists $\epsilon > 0$ such that $(\xi - \epsilon, \xi + \epsilon) \subset G_1$. As $\lim_{n \to \infty} b_n = \xi$, there exists $K \in \mathbb{N}$ such that $b_n \in (\xi - \epsilon, \xi + \epsilon)$ for all $n \geq K$. hence $b_n \in G_1 \cap G_2 \cap I$ for all $n \geq K$ — a contradiction to the fact that $G_1 \cap G_2 \cap I = \emptyset$. Similarly, if $\xi \in G_1$ then we can show that there exists $M \in \mathbb{N}$ such that $a_n \in G_1 \cap G_2 \cap I$ for all $n \geq M$ — again a contradiction.

Thus I is connected. ■

The next theorem states that continuity of function preserves the connectedness of sets.

THEOREM. 2.25 *Suppose that $f : (X, \rho) \to (Y, \sigma)$ be a continuous function and $A \subset X$ be connected. Then $f(A)$ is a connected set in (Y, σ).*

PROOF. If possible, suppose that $f(A)$ is not connected. Then there exists open sets G_1, G_2 in (Y, σ) such that $f(A) \subset G_1 \cup G_2$, $f(A) \cap G_1 \neq \emptyset \neq f(A) \cap G_2$ and $f(A) \cap G_1 \cap G_2 = \emptyset$. This implies that $A \subset f^{-1}(G_1) \cup f^{-1}(G_2)$, $A \cap f^{-1}(G_1) \neq \emptyset \neq A \cap f^{-1}(G_2)$ and $A \cap f^{-1}(G_1) \cap f^{-1}(G_2) = \emptyset$. Since $f^{-1}(G_1), f^{-1}(G_2)$ are open sets this shows that A is not connected — a contradiction.

Thus $f(A)$ is connected. ■

PROBLEM. 2.26 *Prove that if a continuous function $f : \mathbb{R} \to \mathbb{R}$ takes only rational values then it is constant.*

REMARK. 2.27 In view of the above two results we observe that if $f : I \to \mathbb{R}$ is continuous, where I is an interval in $\mathbb{R}$, then $f(I)$ is also an interval in $\mathbb{R}$, which is nothing but the intermediate value theorem of real analysis.

THEOREM. 2.28 *Let (X, ρ) be a metric space $A \subset X$. If A is connected and if $A \subset B \subset \bar{A}$ then B is also connected.*

PROOF. If possible, suppose that B is disconnected. Then there are open sets $G_1, G_2 \subset X$ such that $B \subset G_1 \cup G_2$, $B \cap G_1 \neq \emptyset \neq B \cap G_2$ and $B \cap G_1 \cap G_2 = \emptyset$.

Choose $x \in B \cap G_1$. Then G_1 is a neighbourhood of x and since $x \in \bar{A}$ we have $G_1 \cap A \neq \emptyset$. Similarly $G_2 \cap A \neq \emptyset$. Also since $B \cap G_1 \cap G_2 = \emptyset$ and $A \subset B$ we have $A \cap G_1 \cap G_2 = \emptyset$. This implies that A is disconnected — a contradiction. Hence B must be connected. ■

COROLLARY. 2.29 *If A is a connected subset of a metric space then $\bar{A}$ is also connected.*

This follows by taking $B = \bar{A}$ in the above theorem. ■

DEFINITION. 2.30 Let E be a subset of a metric space (X, ρ). Set

$$\mathcal{A}_E = \{A \subset E : A \text{ is connected } \}.$$

Note that $\mathcal{A}_E$ is partially ordered by the set inclusion '$\subset$'. A maximal element of $(\mathcal{A}_E, \subset)$ is called a *component* of E.

Thus components of a set E are the maximal (w.r.t. set inclusion) connected subsets of E, i.e., $A \subset E$ is a component of E if and only if the following conditions hold:

1. A is a connected subset of E.
2. If $B \subset E$ is connected and $A \subset B$ then $A = B$.

The set E itself is connected if E is the only component of E.

In view of the Corollary 2.29 it follows that

THEOREM. 2.31 *Components of a metric space are closed sets.*

DEFINITION. 2.32 A set $E \subset X$, where (X, ρ) is a metric space, is called *totally disconnected set* if $\{\{x\} : x \in E\}$ is the set of components of E.

PROBLEM. 2.33 *Prove that $\mathbb{Q}$ is totally disconnected.*

THEOREM. 2.34 *Let $\{E_i : i \in I\}$ be a family of connected subsets of a metric space (X, ρ) such that $\cap_{i\in I} E_i \neq \emptyset$. Then $\cup_{i\in I} E_i$ is connected.*

PROOF. If possible suppose that $E = \cup_{i\in I} E_i$ be not connected. Then there exist open sets $G_1, G_2 \subset X$ such that $E \cap G_1 \neq \emptyset \neq E \cap G_2$, $E \subset G_1 \cup G_2$ and $E \cap G_1 \cap G_2 = \emptyset$. Choose $\xi \in \cap_{i\in I} E_i$. Note that $\xi \in E \subset G_1 \cup G_2$. Since $E \cap G_1 \cap G_2 = \emptyset$, either $\xi \in G_1$ or $\xi \in G_2$. Now,

$$\begin{aligned} \xi \in G_1 \quad &\Rightarrow \quad \xi \in G_1 \cap E_i \ \ \forall i \in I \quad \Rightarrow \quad G_1 \cap E_i \neq \emptyset \ \ \forall i \in I \\ &\Rightarrow \quad G_2 \cap E_i = \emptyset \ \ \forall i \in I, \\ &\qquad\qquad \text{since } E_i \cap G_1 \cap G_2 = \emptyset \ \forall i \in I \text{ and each } E_i \text{ is connected .} \end{aligned}$$

Hence $\cup\{E_i \cap G_2 : i \in I\} = \emptyset$, i.e., $\cup\{E_i : i \in I\} \cap G_2 = \emptyset$, i.e., $E \cap G_2 = \emptyset$ — a contradiction.

Similarly, if $\xi \in G_2$ we can show that $E \cap G_1 = \emptyset$ — again a contradiction. Hence E is connected. ■

DEFINITION. 2.35 Let (X, ρ) be a metric space, $E \subset X$ and $x \in E$. Define

$$C(x, E) \quad = \quad \cup\{A \subset E : x \in A, \text{ and } A \text{ is connected}\}.$$

Then by above result $C(x, E)$ is a connected subset of E.

THEOREM. 2.36 *Let (X, ρ) be a metric space, $E \subset X$. Then,*

1. *For all x in E, $C(x,E)$ is a maximal connected subset of E.*
2. *If $C(x,E) \cap C(y,E) \neq \emptyset$ for some $x, y \in E$ then $C(x,E) = C(y,E)$.*
3. *$E = \cup\{C(x,E) : x \in E\}$.*

PROOF.

1. Obviously $C(x,E)$ is a connected subset of E. Let B be a connected subset of E and $x \in E$ such that $C(x,E) \subset B$. So, $x \in B$. Then by definition of $C(x,E)$ we have $B \subset C(x,E)$. Hence $C(x,E) = B$, i.e., $C(x,E)$ is a maximal connected subset of E.
2. Let $x, y \in E$ be such that $C(x,E) \cap C(y,E) \neq \emptyset$. Then $C(x,E) \cup C(y,E)$ is a connected subset of E and also $C(x,E) \subset C(x,E) \cup C(y,E)$. By maximality of $C(x,E)$ we have $C(x,E) = C(x,E) \cup C(y,E)$. Similarly, we can show that $C(y,E) = C(x,E) \cup C(y,E)$. Thus $C(x,E) = C(y,E)$.
3. Immediately follows from the fact $x \in C(x,E) \subset E$. ■

REMARK. 2.37 The above result shows that the set of components of subset of a metric space forms a partition of the subset.

PROBLEM. 2.38 *Let (X,ρ) be a metric space, $E \subset X$. Define a relation '$\sim$' on E by*

$$x \sim y \quad \Longleftrightarrow \quad \text{there exists a connected set } A \subset E \text{ such that } \{x,y\} \subset A.$$

Show that '$\sim$' is an equivalence relation on E and the equivalence class containing x is $C(x,E)$.

2.3 Compactness

DEFINITION. 2.39 Let (X,ρ) be a metric space. A family $\{V_i : i \in I\}$ of open sets in (X,ρ) is said to be an *open cover* of X if $X \subset \cup\{V_i : i \in I\}$. In a similar manner one can define a closed cover or simply a cover.

If $J \subset I$ such that $X \subset \cup\{V_i : i \in J\}$, then $\{V_i : i \in J\}$ is called a *subcover* of $\{V_i : i \in I\}$, if J is a finite set this subcover is called a *finite subcover*.

DEFINITION. 2.40 A metric space (X,ρ) is said to be *compact* if every open cover of it has a finite subcover. If $A \subset X$ then A is called a compact set if (A,ρ_A) is a compact metric space.

EXAMPLE. 2.41

1. Every finite metric space is compact.
2. $\mathbb{R}$ is not compact, $\{(-n,n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$ having no finite subcover.
3. Let X be an infinite set, consider the discrete metric d on X, i.e., $d(x,y) = 1$ if $x \neq y$, $d(x,x) = 0$, for all $x, y \in X$. Let A be an infinite subset of X. Then for any $x \in X$, $S_{\frac{1}{2}}(x) = \{x\}$. Note that $\{S_{\frac{1}{2}}(a) : a \in X\}$ is an open cover of A having no finite subcover. So, A is not compact. However $d(A) = 1$ shows hat A is a bounded set.

THEOREM. 2.42 *Let (X,ρ) be a metric space, $Y \subset X$. Then following are equivalent:*

1. *Y is compact.*
2. *For each family $\{V_i : i \in I\}$ of open sets in (X,ρ), $Y \subset \cup\{V_i : i \in I\}$ implies that there exist $i_1, i_2, \ldots, i_n \in I$ such that $Y \subset V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_n}$.*

PROOF. $1 \Rightarrow 2$: Let Y be compact. Then (Y, ρ_Y) is compact. Let $\{V_i : i \in I\}$ be a family of open sets in (X,ρ) such that $Y \subset \cup\{V_i : i \in I\}$. Put $W_i = V_i \cap Y$ for all $i \in I$. Then each W_i is an open set in the subspace (Y, ρ_Y). Also $Y \subset \cup\{W_i : i \in I\}$. Thus $\{W_i : i \in I\}$ is an open cover of Y. By compactness of Y there exist $i_1, i_2, \ldots, i_n \in I$ such that $Y \subset W_{i_1} \cup W_{i_2} \cup \cdots \cup W_{i_n}$. Since $W_i \subset V_i$ for all $i \in I$, it follows that $Y \subset V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_n}$.

$2 \Rightarrow 1$: Assume 2 holds. Let $\{V_i : i \in I\}$ be an open cover of Y in (Y, ρ_Y). Since for each i in I, V_i is open in (Y, ρ_Y) there exists open set W_i in (X,ρ) such that $V_i = W_i \cap Y$. So, $Y \subset \cup\{W_i : i \in I\}$. By 2 there exist $i_1, i_2, \ldots, i_n$ in I such that $Y \subset W_{i_1} \cup W_{i_2} \cup \cdots \cup W_{i_n}$. This implies that $Y \subset V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_n}$. Hence Y is compact. ■

The next two results relates continuity with compactness.

THEOREM. 2.43 *Let $f : (X,\rho) \to (Y,\sigma)$ be a continuous function. If $A \subset X$ is a compact set then $f(A) \subset Y$ is also a compact set.*

PROOF. Let $\{V_i : i \in I\}$ be a family of open sets in (Y,σ) such that $f(A) \subset \cup\{V_i : i \in I\}$. Then $A \subset f^{-1}(\cup\{V_i : i \in I\}) = \cup\{f^{-1}(V_i) : i \in I\}$ Since f is continuous $f^{-1}(V_i)$ is open in (X,ρ) for each $i \in I$. By compactness of A there exist $i_1, i_2, \ldots, i_n \in I$ such that $A \subset f^{-1}(V_{i_1}) \cup f^{-1}(V_{i_2}) \cup \cdots \cup f^{-1}(V_{i_n})$. This implies that $f(A) \subset V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_n}$. Hence $f(A)$ is compact. ■

THEOREM. 2.44 *Let $f : (X,\rho) \to (Y,\sigma)$ be a continuous function. If (X,ρ) is a compact metric space then f is uniformly continuous.*

PROOF. Let $\epsilon > 0$. For any x in X, since f is continuous at x, there exists $\delta(x) > 0$ such that for all y in X,

$$y \in S_{\delta(x)}(x) \Rightarrow \sigma(f(x), f(y)) < \frac{\epsilon}{2} \tag{1}$$

Note that $\{S_{\frac{\delta(x)}{2}}(x) : x \in X\}$ is an open cover of X. By compactness of X there are $x_1, x_2, \ldots, x_n$ in X such that

$$X \subset S_{\frac{\delta(x_1)}{2}}(x_1) \cup S_{\frac{\delta(x_2)}{2}}(x_2) \cup \cdots \cup S_{\frac{\delta(x_n)}{2}}(x_n).$$

Set $\delta = \min\{\frac{\delta(x_1)}{2}, \frac{\delta(x_2)}{2}, \ldots, \frac{\delta(x_n)}{2}\}$. Clearly $\delta > 0$. Let $x, y \in X$ such taht $\rho(x, y) < \delta$. Then there exists $i = 1, 2, \ldots, n$ such taht $x \in S_{\frac{\delta(x_i)}{2}}(x_i) \subset S_{\delta(x_i)}(x_i)$. Also $\rho(x_i, y) \leq \rho(x_i, x) + \rho(x, y) < \frac{\delta(x_i)}{2} + \delta \leq \frac{\delta(x_i)}{2} + \frac{\delta(x_i)}{2} = \delta(x_i)$. Thus $y \in S_{\delta(x_i)}(x_i)$. Hence By (1),

$$\sigma(f(x), f(y)) \leq \sigma(f(x), f(x_i)) + \sigma(f(x_i), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus f is uniformly continuous. ■

THEOREM. 2.45 *Every compact subset of a metric space is closed.*

PROOF. Let (X, ρ) be a be a metric space, $A \subset X$ be compact. Choose $x \in X - A$ It is sufficient to show that x is not a limit point of A. Now, for all $a \in A$, $\rho(x, a) > 0$. Let $\delta(a) = \frac{1}{2}\rho(x, a)$. Then $\{S_{\delta(a)}(a) : a \in A\}$ is an open cover of A. Since A is compact there exist $a_1, a_2, \ldots, a_n \in A$ such taht $A \subset S_{\delta(a_1)}(a_1) \cup S_{\delta(a_2)}(a_2) \cup \cdots \cup S_{\delta(a_n)}(a_n)$. Let $\delta = \min\{\delta(a_1), \delta(a_2), \ldots, \delta(a_n)\}$. Since $S_{\delta(a_i)}(x) \cap S_{\delta(a_i)}(a_i) = \emptyset$ for all $i = 1, 2, \ldots, n$ it follows that $S_\delta(x) \cap S_{\delta(a_i)}(a_i) = \emptyset$ for all $i = 1, 2, \ldots, n$ and hence $S_\delta(x) \cap A = \emptyset$. Thus x is not a limit point of A. So A is closed. ■

THEOREM. 2.46 *Every closed subset of a compact metric space is compact.*

PROOF. Let (X, ρ) be a metric space, $A \subset X$ be closed. Let $\{V_i : i \in I\}$ be a family of open sets covering A. Let $V = X - A$. Then V is an open set, so the family $\{V\} \cup \{V_i : i \in I\}$ is an open cover of X. By compactness of X this cover has a finite subcover, say $V, V_{i_1}, V_{i_2}, \ldots, V_{i_n}$. Thus $X \subset V \cup V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_n}$. This implies that $A \subset V \cup V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_n}$. Since $A \cap V = \emptyset$, $A \subset V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_n}$. Hence A is compact. ■

DEFINITION. 2.47 A subset A of a metric space (A, ρ) is said to be a *totally bounded set* if for any $\epsilon > 0$ there exist $x_1, x_2, \ldots, x_n$ in X such that $A \subset S_\epsilon(x_1) \cup S_\epsilon(x_2) \cup \cdots \cup S_\epsilon(x_n)$.

It can be observed that a subset of a totally bounded subset is totally bounded. It can also be observed that every totally bounded set is bounded, however the converse is not true. There are sets in metric spaces which are bounded but not totally bounded.

EXAMPLE. 2.48 Consider any infinite set X with discrete metric d. Let A be an infinite subset of X. If we take $\epsilon = \frac{1}{2}$ then for any $x \in X$, $S_\epsilon(x) = \{x\}$. Hence it is impossible to find a finite number of points $x_1, x_2, \ldots, x_n$ in A for which $A \subset S_\epsilon(x_1) \cup S_\epsilon(x_2) \cup \cdots \cup S_\epsilon(x_n)$. Thus A is totally bounded. However $d(A) = 1$ shows that A is bounded.

THEOREM. 2.49 *Let A be a subset of a metric space (X, ρ). Then A is totally bounded if and only if for any $\epsilon > 0$ there exist finitely many subsets $A_1, A_2, \ldots, A_n$ of A such that $A = A_1 \cup A_2 \cup \cdots \cup A_n$ and $\rho(A_k) \leq \epsilon$ for all $k = 1, 2, \ldots, n$.*

PROOF. Let A be totally bounded and $\epsilon > 0$. Then there exists $x_1, x_2, \ldots, x_n \in A$ such that $A \subset S_{\epsilon/2}(x_1) \cup S_{\epsilon/2}(x_2) \cup \cdots \cup S_{\epsilon/2}(x_n)$. Put $A_k = A \cap S_{\epsilon/2}(x_k)$ for $k = 1, 2, \ldots, n$. Then $\rho(A_k) \leq \epsilon$ for all $k = 1, 2, \ldots, n$ and $A = A_1 \cup A_2 \cup \cdots \cup A_n$.

Convversely, let the condition hold and $\epsilon > 0$. Choose a real number δ such that $0 < \delta < \epsilon$. By the condition there exists $A_1, A_2, \ldots, A_n \subset A$ such taht $\rho(A_i) \leq \delta$ for all $i = 1, 2, \ldots, n$ and $A = A_1 \cup A_2 \cup \cdots \cup A_n$. Choose $x_i \in X$ such that $A_i \subset S_\epsilon(x_i)$ for all $i = 1, 2, \ldots, n$. Thus $A \subset \cup\{S_\epsilon(x_i) : i = 1, 2, \ldots n\}$, i.e., A is totally bounded. ■

THEOREM. 2.50 *A subset of a metric space is totally bounded if and only if every sequence in it has a Cauchy subsequence.*

PROOF. Let (X, ρ) be a metric space, $A \subset X$. Assume that A is totally bounded. Let $\{x_n\}$ be a sequence in A. Choose subsets $A_{11}, A_{12}, \ldots, A_{1n_1}$ of A such that $\rho(A_{1i}) \leq 1$ for $i = 1, 2, \ldots, n_1$ and $A = A_{11} \cup A_{12} \cup \ldots \cup A_{1n_1}$. Then there exists $i \in \{1, 2, \ldots, n_1\}$ such that A_{1i} contains x_n for infinitely many n. We denote this set by B_1, then B_1 is totally bounded. Again choose subsets $A_{21}, A_{22}, \ldots, A_{2n_2}$ of B_1 such that $\rho(A_{2i}) \leq \frac{1}{2}$ for all $i \in \{1, 2, \ldots, n_2\}$ and $B_1 = A_{21} \cup A_{22} \cup \ldots \cup A_{2n_2}$. Now one of the sets $A_{21}, A_{22}, \ldots, A_{2n_2}$ contains x_n for infinitely many n in $\mathbb{N}$. Call this set B_2. Assume that we have found $B_1, B_2, \ldots, B_k$ such taht $A \supset B_1 \supset \cdots \supset B_k$, $\rho(B_i) \leq \frac{1}{i}$ for all $i = 1, 2, \ldots, k$ and each B_i contains x_n for infinitely many n. By arguments similar to those used above, we can find $B_{k+1} \subset B_k$ such that $\rho(B_{k+1}) \leq \frac{1}{k+1}$ and B_{k+1} contains x_n for infinitely many n. So, by induction there exist subsets $B_1, B_2, \ldots$ of A such that

1. $A \supset B_1 \supset B_2 \supset \cdots$.
2. $\rho(B_k) \leq \frac{1}{k}$ for all $k \in \mathbb{N}$.
3. Each B_k contains x_n for infinitely many n.

Now choose a positive integer n_1 such that $x_{n_1} \in B_1$. Suppose we have chosen integers $n_1 < n_2 < \cdots < n_k$ such that $x_{n_i} \in B_i$ for $i = 1, 2, \ldots, k$. Since B_{k+1} contains infinitely many x_n we can find a positive integer $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in B_{k+1}$. So, again by induction we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in B_k$ for all $k \in \mathbb{N}$.

Let $\epsilon > 0$. Choose a positive integer m such that $\frac{1}{m} < \epsilon$. Whenever $i, j > m$, $x_{n_i}, x_{n_j} \in B_m$, since $\rho(B_m) \leq \frac{1}{m}$, $\rho(x_{n_i}, x_{n_j}) \leq \frac{1}{m} < \epsilon$. Therefore, $\{x_{x_k}\}$ is a Cauchy subsequence of $\{x_n\}$.

Conversely, suppose that every sequence in A has a Cauchy subsequence. If possible, suppose that A is not totally bounded. Then there exists $\epsilon > 0$ such that for all finitely many points $x_1, x_2, \ldots, x_n$ in A, $A \not\subset S_\epsilon(x_1) \cup S_\epsilon(x_2) \cup \cdots \cup S_\epsilon(x_n)$. Choose $x_1 \in A, x_2 \in A - S\epsilon(x_1)$. Suppose that we have found $x_1, x_2, \ldots, x_n \in A$ such taht $x_i \in A - S_\epsilon(x_1) \cup S_\epsilon(x_2) \cup \cdots \cup S_\epsilon(x_i)$ for all $i = 1, 2, \ldots, n$. Since $A \not\subset S_\epsilon(x_1) \cup S_\epsilon(x_2) \cup \cdots \cup S_\epsilon(x_n)$, we can find $x_{n+1} \in A - S_\epsilon(x_1) \cup S_\epsilon(x_2) \cup \cdots \cup S_\epsilon(x_n)$. Thus by induction we get a sequence $\{x_n\}$ in A such that for all $i \geq 2$, $x_i \notin S_\epsilon(x_1) \cup S_\epsilon(x_2) \cup \cdots \cup S_\epsilon(x_{i-1})$. Let $i, j \in \mathbb{N}$. Then, if $j > i$, then $x_j \notin S_\epsilon(x_i)$ and hence $\rho(x_i, x_j) \geq \epsilon$. This shows that $\{x_n\}$ can not have a Cauchy subsequence — a contradiction. ■

DEFINITION. 2.51 A subset A of a metric space is said to be *sequentially compact* if each sequence in A has a convergent subsequence with limit in A.

THEOREM. 2.52 *A subset A of a metric space is sequentially compact if and only if every sequence in A has a cluster point in A.*

PROOF. Proof is easy.

DEFINITION. 2.53 Let A be a subset of a metric space (X, ρ) and $\mathcal{A}$ be a family of open sets of (X, ρ) covering A. Then $\delta > 0$ is said to be a *Lebesgue number* of the open cover $\mathcal{A}$ for A if for all $B \subset A$, $\rho(B) < \delta$ implies that there exists $V \in \mathcal{A}$ such that $B \subset V$

THEOREM. 2.54 *Let A be a sequentially compact subset of a metric space (X, ρ) and $\mathcal{A}$ be a family of open sets covering A. Then $\mathcal{A}$ has a Lebesgue number.*

PROOF. If possible suppose that $\mathcal{A}$ has no Lebesgue number. Then for each positive integer n there exists $A_n \subset A$ such that $\rho(A_n) < \frac{1}{n}$ and $A_n \not\subset G$ for all $G \in \mathcal{A}$. By induction we can choose a sequence $\{a_n\}$ in A such that $a_n \in A_n$ for all $n \in \mathbb{N}$. Since A is sequentially compact, $\{a_n\}$ has a cluster point, say l, in A. Since $\mathcal{A}$ covers A, there exists $G \in \mathcal{A}$ such that $l \in G$. G being open, we can choose $\epsilon > 0$ such that $S_{2\epsilon}(l) \subset G$. Also we can choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ and $\rho(a_n, l) < \epsilon$. Now, let $x \in A_n$. Then $\rho(x, a_n) \leq \rho(A_n) < \frac{1}{n} < \epsilon$ and hence $\rho(x, l) \leq \rho(x, a_n) + \rho(a_n, l) < \epsilon + \epsilon = 2\epsilon$. Thus $x \in S_{2\epsilon}(l)$ which shows that $A_n \subset S_{2\epsilon}(l) \subset G$ — a contradiction. ■

PROBLEM. 2.55 *Prove that each sequentially compact metric space is totally bounded.*

DEFINITION. 2.56 Let A be a subset of a metric space (X, ρ). Then A is said to have *Bolzano-Weierstrass (BW) property* if each infinite subset of A has a limit point in A.

It is vacuously true that every finite set has the BW property.

THEOREM. 2.57 *Let (X, ρ) be a metric space, $A \subset X$. Then the followings are equivalent:*

1. *A is compact.*

2. *A has the BW property.*

3. *A is sequentially compact.*

PROOF. $1 \Rightarrow 2$: Assume A is a compact set and B be an infinite subset of A. If possible, suppose that B has no limit point. Then each $a \in A$ is not a limit point of B. So, for all $a \in A$ there exists $\delta(a) > 0$ such taht $S_{\delta(a)}(a) \cap B \subset \{a\}$. The family $\{S_{\delta(a)}(a) : a \in A\}$ is an open cover of A and by compactness of A it has a finite subcover, i.e., there exist $a_1, a_2, \ldots, a_n \in A$ such that $A \subset S_{\delta(a_1)}(a_1) \cup S_{\delta(a_2)}(a_2) \cup \cdots \cup S_{\delta(a_n)}(a_n)$. So $B \subset S_{\delta(a_1)}(a_1) \cup S_{\delta(a_2)}(a_2) \cup \cdots \cup S_{\delta(a_n)}(a_n)$, i.e., $B \subset (S_{\delta(a_1)}(a_1) \cap B) \cup (S_{\delta(a_2)}(a_2) \cap B) \cup \cdots \cup (S_{\delta(a_n)}(a_n) \cap B) \subset \{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_n\} = \{a_1, a_2, \ldots, a_n\}$ — a contradiction as B is an infinite set.

$2 \Rightarrow 3$: Assume that A has the BW property. Let $\{x_n\}$ be a sequence in A, R be the range of $\{x_n\}$, i.e., $R = \{x_n : n \in \mathbb{N}\}$. Two cases may arise:

Case 1: R is finite. Then there exists $\xi \in A$ such that $x_n = \xi$ for infinitely many $n \in \mathbb{N}$ and in this case ξ is a cluster poin of $\{x_n\}$.

Case 2: R is infinite. Then, since A has the BW property, R has a limit point, say ξ, in A. Clearly ξ is a cluster point of $\{x_n\}$.

Hence A is sequentially compact.

$3 \Rightarrow 1$: Assume A is sequentially compact. Let $\mathcal{G} = \{G_i : i \in I\}$ be an open cover of A. Since A is sequentially compact, $\mathcal{G}$ has a Lebesgue bumber $\delta > 0$. Also A is totally bounded, hence there exist $a_1, a_2, \ldots, a_n \in A$ such that $A \subset S_{\frac{\delta}{3}}(a_1) \cup S_{\frac{\delta}{3}}(a_2) \cup \cdots \cup S_{\frac{\delta}{3}}(a_n)$. Note that for all $k = 1, 2, \ldots, n$, $\rho(A \cap S_{\frac{\delta}{3}}(a_k)) \le \frac{2\delta}{3} < \delta$ and hence there exists $G_{i_k} \in \mathcal{G}$ such that $A \cap S_{\frac{\delta}{3}}(a_k) \subset G_{i_k}$. Thus,

$$
\begin{aligned}
A &= (A \cap S_{\frac{\delta}{3}}(a_1)) \cup (A \cap S_{\frac{\delta}{3}}(a_2)) \cup \cdots \cup (A \cap S_{\frac{\delta}{3}}(a_n)) \\
&\subset G_{i_1} \cup G_{i_2} \cup \cdots \cup G_{i_n}.
\end{aligned}
$$

Hence A is compact. ■

THEOREM. 2.58 *A subset of a metric space is compact if and only if it is complete and totally bounded.*

PROOF. Let (X, ρ) be a metric space, $A \subset X$ be compact. For any $\epsilon > 0$, $\{S_\epsilon(a) : a \in A\}$ is an open cover of A and hence by compactness of A there exist $a_1, a_2, \ldots, a_n$ in A such that $\{S_\epsilon(a_i) : 1 \leq i \leq n\}$ covers A. Thus A is totally bounded. Let $\{a_n\}$ be a Cauchy sequence in A. Since A is compact, it is sequentially compact and hence $\{a_n\}$ has a cluster point. Recall that if a cauchy sequence has a cluster point then it is convergent. So $\{a_n\}$ is convergent and hence A is complete.

Conversely, let A be complete and totally bounded and $\{a_n\}$ be a sequence in A. Since A is totally bounded $\{a_n\}$ has a Cauchy subsequence, say $\{a_{n_k}\}$, and by completenes of A, $\{a_{n_k}\}$ is convergent. Thus every sequence in A has a convergent subsequence. So, A is sequentially compact and hence is compact. ■

Problem. 2.59 *Prove that every bounded subset of $\mathbb{R}$ is totally bounded.*

Solution: Let $A \subset \mathbb{R}$ be bounded. then there exists $a, b \in \mathbb{R}, a, b$, such that $A \subset [a, b]$. Let $\epsilon > 0$ be arbitrary. Let n be the smallest positive integer such taht $\frac{b-a}{n} < \frac{\epsilon}{2}$. Choose $\delta > 0$ such that $\delta < \frac{\epsilon}{2}$. Let for all $k = 1, 2, \ldots, n$, $I_k = [a + (k-1)\delta, a + k\delta]$. Then $[a, b] \subset \cup_{k=1}^n I_k$, i.e., $A \subset \cup_{k=1}^n I_k$. Also $\rho(I_k) = 2\delta < \epsilon$. Thus A is totally bounded.

Problem. 2.60 *Prove that every bounded subset of $\mathbb{R}^n$, where $n \in \mathbb{N}$, is totally bounded.*

Solution: Let A be a bounded subset of $\mathbb{R}^n$ and $\epsilon > 0$ be a real number. For $i = 1, 2, \ldots, n$ let $A_i = \pi_i(A)$. Then $A \subset A_1 \times A_2 \times \cdots \times A_n$ and each A_i is bounded subset of $\mathbb{R}$, and hence each A_i is totally bounded. So for each $i \in \{1, 2, \ldots, n\}$, there exists $A_{i1}, A_{i2}, \ldots, A_{im_i}$ such that $A_i \subset A_{i1} \cup A_{i2} \cup \cdots \cup A_{im_i}$ and $\rho(A_{ik}) < \epsilon/\sqrt{n}$ for all $k \in \{1, 2, \ldots, m_i\}$.

Now, for all $(p_1, p_2, \ldots, p_n) \in \{1, 2, \ldots, m_1\} \times \{1, 2, \ldots, m_2\} \times \cdots \times \{1, 2, \ldots, m_n\}$, set

$$A_{p_1 p_2 \cdots p_n} = A_{1p_1} \times A_{2p_2} \times \cdots \times A_{np_n}.$$

Then for any $x, y \in A_{p_1 p_2 \cdots p_n}$,

$$\rho(x, y) = \sqrt{\sum_{i=1}^n (\pi_i(x) - \pi_i(y))^2} < \sqrt{\sum_{i=1}^n (\epsilon/\sqrt{n})^2} = \epsilon$$

which implies that $\rho(A_{p_1 p_2 \cdots p_n}) \leq \epsilon$.

Also $A \subset \cup\{A_{p_1 p_2 \cdots p_n} : (p_1, p_2, \ldots, p_n) \in \{1, 2, \ldots, m_1\} \times \{1, 2, \ldots, m_2\} \times \cdots \times \{1, 2, \ldots, m_n\}$. Since this collection is finite, A is totally bouonded.

As a consequence of the above results we have the following.

Theorem. 2.61 *For $n \in \mathbb{N}$, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Let A be a closed and bounded subset of $\mathbb{R}^n$. Then A is totally bounded. Also since a subset of $\mathbb{R}^n$ is is closed if and only if it is complete, it follows that A is complete. Thus A is complete and totally bounden, i.e., A is compact.

The converse part is immediate. ■

Theorem. 2.62 (Bolzano-Weierstrass Theorem): *For any integer $n \geq 1$, every infinite bounded subset of $\mathbb{R}^n$ has a limit point.*

Proof. Let A be an infinite bounded subset of $\mathbb{R}^n$. Then $\bar{A}$ is also bounded. Since $\bar{A}$ is closed, it is compact. So $\bar{A}$ has the B-W property. A being an infinite subset of $\bar{A}$, A has a limit point (in $\bar{A}$). ■

Theorem. 2.63 *let A be a compact subset of a metric space (X, ρ) and $f : A \to \mathbb{R}$ is a continuous function. Then f is bounded and attains its bounds.*

Proof. As $f(A)$ is a compact subset of $\mathbb{R}$, it is closed and bounded. So f is bounded. If $m = \inf f(A), M = \sup f(A)$ then $m, M \in f(A)$. So there exists $a, b \in A$ such that $f(a) = m, f(b) = M$. Hence f attains its bounds. ■

2.4 Contraction Map and Banach's Contraction Principle

Definition. 2.64 Let (X, ρ) be a metric space and $f : X \to X$ be a function. Then f is said to be a *contraction mapping* if there exists $r \in \mathbb{R}, 0 \leq r < 1$ such that

$$\rho(f(x), f(y)) \leq r \cdot \rho(x, y) \ \ \forall x, y \in X.$$

It immediately follows that each contraction function satisfies Lipschitz's condition and hence it is a uniformly continuous function.

Definition. 2.65 Let X be a set $f : X \to X$ be a function. A point $a \in X$ is said to be a *fixed point* of f if $f(a) = a$.

Theorem. 2.66 [Banach's contraction principle] *Let f be a contraction mapping on a complete metric space (X, ρ). Then f has a unique fixed point.*

Proof. Choose $x_0 \in X$ arbitrarily. Define inductive a sequence $\{x_n\}$ in X as follows: $x_1 = f(x_0), x_2 = f(x_1), \ldots x_{n+1} = f(x_n)$ for all $n \geq 2$.

Since f is a contraction mapping there exists $r \in \mathbb{R}$, $0 \leq r < 1$, such that $\rho(f(x), f(y)) \leq r.\rho(x, y)$ for all $x, y \in X$. Let $k \in \mathbb{N}$. Then

$$\begin{aligned}\rho(x_k, x_{k-1}) &= \rho(f(x_{k-1}), f(x_{k-2})) \le r.\rho(x_{k-1}, x_{k-2}) \le r^2.\rho(x_{k-2}, x_{k-3}) \\ &\le \cdots \le r^{k-1}.\rho(x_1, x_0).\end{aligned}$$

Hence for $m, n \in \mathbb{N}, m > n$,

$$\begin{aligned}\rho(x_m, x_n) &\le \rho(x_m, x_{m-1}) + \rho(x_{m-1}, x_{m-2}) + \cdots + \rho(x_{n+1}, x_n) \\ &\le r^{m-1}\rho(x_1, x_0) + r^{m-2}\rho(x_1, x_0) + \cdots + r^n\rho(x_1, x_0) \\ &= r^n.\rho(x_1, x_0)[1 + r + r^2 + \cdots + r^{m-1-n}] \\ &= r^n.\rho(x_1, x_0)\frac{1 - r^{m-n}}{1 - r} \le \rho(x_1, x_0)\frac{r^n}{1 - r}.\end{aligned}$$

Since $|r| < 1$, $\lim_{n\to\infty} r^n = 0$ and hence $\lim_{m,n\to\infty} \rho(x_m, x_n) = 0$. Thus $\{x_n\}$ is a Cauchy sequence in (X, ρ). By completeness of (X, ρ) there exists $y_0 \in X$ such that $\lim_{n\to\infty} x_n = y_0$. Since f is continuous, $\lim_{n\to\infty} f(x_n) = f(y_0)$, i.e., $\lim_{n\to\infty} x_{n+1} = f(y_0)$ and hence $y_0 = f(y_0)$. Thus y_0 is a fixed point of f.

If possible, let there be two fixed points, say y_0 and y_0'. Then

$$\rho(y_0, y_0') = \rho(f(y_0), f(y_0')) \le r.\rho(y_0, y_0').$$

This is impossible, since $0 \le r < 1$, unless $y_0 = y_0'$. Thus f has unique fixed point. ■

We conclude this note with an applications of Banach's Contraction Principle; viz., Picard's Theorem on existance of unique solution of differential equation.

2.4.1 Picard's Theorem

THEOREM. 2.67 *Let f be a real valued function defined on the rectangle $R = [a_1, a_2] \times [b_1, b_2]$. Suppose that f and $\frac{\partial f}{\partial y}$ are continuous on R and (x_0, y_0) is an interior point on R. Then the differential equation $\frac{dy}{dx} = f(x, y)$ has a unique solution $y = g(x)$ such that $y_0 = g(x_0)$.*

PROOF. Note that $y = g(x)$ is a solution of $\frac{dy}{dx} = f(x, y)$ for all x in a neighbourhood N_{x_0} of x_0 satisfying $y_0 = g(x_0)$ if and only if

$$g(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt \quad \forall x \in N_{x_0}.$$

To complete the proof it is sufficient to show that that there exists a unique $g(x)$ satisfying the above condition.

Since f and $\frac{\partial f}{\partial y}$ are continuous on the compact set $[a_1, a_2] \times [b_1, b_2]$ there exists a real $M > 0$ such that

$$|f(x,y)| \leq M \quad \text{and} \quad \left|\frac{\partial}{\partial y} f(x,y)\right| \leq M \tag{2}$$

for all $(x, y) \in [a_1, a_2] \times [b_1, b_2]$. Let x be an arbitrary but fixed real chosen in $[a_, a_2]$ and $y_1, y_2 \in [b_1, b_2]$. Then by Lagrange's Mean Value Theorem there exists $\theta, 0 < \theta < 1$, such that

$$f(x, y_1) - f(x, y_2) \;=\; (y_2 - y_1) \frac{\partial}{\partial y} f(x, y_2 + \theta(y_1 - y_2)),$$

Hence,

$$|f(x, y_1) - f(x, y_2)| \;\leq\; |y_1 - y_2| M. \tag{3}$$

Thus for a fixed x, f satisfies the Lipschitz condition with respect to the variable y. Choose a positive real α such that $M\alpha < 1$ and $[x_0 - \alpha,\, x_0 + \alpha] \times [y_0 - M\alpha,\, y_0 + M\alpha] \subset [a_1, a_2] \times [b_1, b_2]$.

Note that $C([x_0 - \alpha,\, x_0 + \alpha])$, the set of all real valued continuous functions defined on $[x_0 - \alpha,\, x_0 + \alpha]$, is a real vector space, moreover it is a complete normed linear space with respect to supnorm.

Define a subset X of $C([x_0 - \alpha,\, x_0 + \alpha])$ by

$$g \in X \quad \Longleftrightarrow \quad \|g - \underline{\mathbf{y_0}}\| \leq M\alpha,$$

where $\underline{\mathbf{y_0}}$ denotes the constant function defined by $\underline{\mathbf{y_0}}(x) = y_0$ for all $x \in [x_0 - \alpha,\, x_0 + \alpha]$, i.e., X is the closed ball $S_{M\alpha}[\underline{\mathbf{y_0}}]$ in $C([x_0 - \alpha,\, x_0 + \alpha])$ with centre at $\underline{\mathbf{y_0}}$ and radius $M\alpha$. So, X is a closed subspace of $C([x_0 - \alpha,\, x_0 + \alpha])$. Since $C([x_0 - \alpha,\, x_0 + \alpha])$ is complete and X is a closed subspace of it, X is a complete metric space with respect to metric induced by the norm of $C([x_0 - \alpha,\, x_0 + \alpha])$.

Now, for $g \in X, t \in [x_0 - \alpha,\, x_0 + \alpha]$, since $|g(t) - y_0| < M\alpha$, it follows that $(t, g(t))$ belongs to $[x_0 - \alpha,\, x_0 + \alpha] \times [y_0 - M\alpha,\, y_0 + M\alpha]$. Define a mapping $T(g) : [x_0 - \alpha,\, x_0 + \alpha] \to \mathbb{R}$ by,

$$T(g)(x) \;=\; y_0 + \int_{x_0}^{x} f(t, g(t))\, dt, \;\; \forall t \in [x_0 - \alpha,\, x_0 + \alpha].$$

Clearly, $T(g) \in C([x_0 - \alpha,\, x_0 + \alpha])$. Also for all $x \in [x_0 - \alpha,\, x_0 + \alpha]$,

$$|T(g)(x) - y_0| \;=\; \left|\int_{x_0}^{x} f(t, g(t))\, dt\right| \;\leq\; \int_{x_0}^{x} |f(t, g(t))|\, dt \;\leq\; M\alpha.$$

So, $\sup\{|T(g)(x) - y_0| : x \in [x_0 - \alpha,\, x_0 + \alpha]\} \le M\alpha$ and hence $\|T(g) - \underline{\mathbf{y_0}}\| \le M\alpha$. Thus $T(g) \in X$, i.e., T is a mapping from X to X.

Let $g_1, g_2 \in X$ and $x \in [x_0 - \alpha,\, x_0 + \alpha]$. Then

$$\begin{aligned} |T(g_1)(x) - T(g_2)(x)| &= \left| \int_{x_0}^{x} [f(t, g_1(t)) - f(t, g_2(t))] \ dt \right| \\ &\le M|g_1(t) - g_2(t)| \cdot |x - x_0| \quad \text{(by 3)} \\ &\le M|g_1(t) - g_2(t)|\alpha. \end{aligned}$$

Thus $\|T(g_1) - T(g_2)\| \le M\alpha\|g_1 - g_2\|$.

Since $M\alpha < 1$ it follows that $T : X \to X$ is a contraction mapping. Since X is a complete metric space there exists a unique $g \in X$ such that $T(g) = g$, i.e., $T(g)(x) = g(x)$ for all x in $[x_0 - \alpha,\, x_0 + \alpha]$. Hence

$$g(x) = y_0 + \int_{x_0}^{x} f(t, g(t)) \ dt \ \ \forall t \in [x_0 - \alpha,\, x_0 + \alpha].$$

This completes the proof. ∎