Study Material on Fourier Series

Department of Mathematics, P. R. Thakur Govt. College
MTMACOR08T: Unit 4 (Semester - 4)

Syllabus

Unit 4: Fourier series: Definition of Fourier coefficients and series, Reimann Lebesgue lemma, Bessel's inequality, Parseval's identity, Dirichlet's condition. Examples of Fourier expansions and summation results for series.

Motivation

Consider the vector space $\mathbb{R}^n$ and it standard basis $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, $\dots$, $e_n = (0, 0, 0, \dots, 1)$. For $n = 3$, e_1, e_2, e_3 are usually written as $\hat{i}, \hat{j}, \hat{k}$ respectively. A vector $x = (x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$ can be written as $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$, as a linear combination of the basis vectors. It can be observed that the coefficients $x_1, x_2, \dots, x_n$ are obtained as $x_k = x \cdot e_k, 1 \le k \le n$, where the dot product '$\cdot$' is defined by, for $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$, $x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n$. Here another things to observe that (i) the basis vectors are *normalised*, i.e., $\|e_k\| = \sqrt{e_k \cdot e_k} = 1$ and (ii) they are *orthogonal* to each other, i.e., $e_j \cdot e_k = 0$ whenever $j \ne k$.

The concept can be generalized for an arbitrary vector space V where an *inner product* is defined (called an *inner product space*). Here, for the time being, we consider only finite dimensional vector spaces over the field of real numbers. A function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ is called an *inner product* if (i) $\langle u, v\rangle = \langle v, u\rangle$, (ii) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$, (iii) $\langle cu, v\rangle = c\langle u, v\rangle$ and (iv) $\langle u, u\rangle \ge 0$, where $u, v, w \in V$ and $c \in R$. It can be observed that the dot product is an example of inner product, $\langle u, v\rangle = u \cdot v$, $u, v \in \mathbb{R}^n$. Two vectors u, v are called *orthogonal* if $\langle u, v\rangle = 0$. For a vector $u \in V$, its *norm* is defined as $\|u\| = \sqrt{\langle u, u\rangle}$ (where in $\mathbb{R}^3$ it is called the *modulus* of the vector).

Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an orthogonal basis of V (such a basis exists – it can be proved), i.e., $\langle \alpha_i, \alpha_j\rangle = 0$ for $i \ne j, 1 \le i \le j \le n$. Then an arbitrary vector $\alpha \in V$ can be expressed as

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$ where $c_1, c_2, \ldots, c_n$ are scalars. Taking inner product with α_i on both sides we have $\langle \alpha, \alpha_i\rangle = \langle c_1\alpha_1, \alpha_i\rangle + \langle c_2\alpha_2, \alpha_i\rangle + \cdots + \langle c_n\alpha_n, \alpha_i\rangle = c_1\langle \alpha_1, \alpha_i\rangle + c_2\langle \alpha_2, \alpha_i\rangle + \cdots + c_n\langle \alpha_n, \alpha_i\rangle = c_i\langle \alpha_i, \alpha_i\rangle = c_i\|\alpha_i\|^2$. Hence the scalars c_i are calculated as $c_i\ \|\alpha_i\|^2 = \langle \alpha, \alpha_i\rangle$ or, $c_i = \frac{1}{\|\alpha_i\|^2}\langle \alpha, \alpha_i\rangle$, $i = 1, 2, \ldots, n$. Thus, we can write

$$\alpha = \sum_{i=1}^{n} \frac{\langle \alpha, \alpha_i\rangle}{\|\alpha_i\|^2}\, \alpha_i = \frac{\langle \alpha, \alpha_1\rangle}{\|\alpha_1\|^2}\, \alpha_1 + \frac{\langle \alpha, \alpha_2\rangle}{\|\alpha_2\|^2}\, \alpha_2 + \cdots + \frac{\langle \alpha, \alpha_n\rangle}{\|\alpha_n\|^2}\, \alpha_n.$$

When we try to extend this result to an infinite dimensional inner product spaces, the sum will be an infinite series and the question of convergence will arise. Under certain condition (completeness, totality etc.) the series converges and we can write in such a space V, $\alpha \in V$, $\alpha = \sum_{n=1}^{\infty} c_n\alpha_n$, where $\{\alpha_n : n \in \mathbb{N}\}$ is an orthogonal basis for V and the scalars c_n are calculated as $c_n = \frac{\langle \alpha, \alpha_n\rangle}{\langle \alpha_n, \alpha_n\rangle}$, $n \in \mathbb{N}$.

Definition and Determination of Fourier Series

It can be verified that the set $R([a, b])$ of all the integrable functions defined on an interval $[a, b]$ is a vector space over the field of real numbers. An inner product on $R([a, b])$ can be defined as $\langle f, g\rangle = \displaystyle\int_a^b f(x)g(x)\,\mathrm{d}x$, for all $f, g \in R([a, b])$.

The orthogonal relations of the trigonometric functions $\sin nx, \cos nx$, $m, n \in \mathbb{N}$, in the interval $[-\pi, \pi]$ are as follows:

Theorem. 1 *For $m, n \in \mathbb{N}$,*

$$\text{(i)} \int_{-\pi}^{\pi} \sin mx\,\mathrm{d}x = \int_{-\pi}^{\pi} \cos mx\,\mathrm{d}x = 0 \qquad \textit{and} \qquad \text{(ii)} \int_{-\pi}^{\pi} \sin mx \cos nx\,\mathrm{d}x = 0.$$

Proof. Easy verification. ■

Theorem. 2 *For $m, n \in \mathbb{N}$,*

$$\text{(i)} \int_{-\pi}^{\pi} \sin mx \sin nx\,\mathrm{d}x \begin{array}{l} = 0,\ m \neq n \\ = \pi,\ m = n \end{array} \qquad \text{(ii)} \int_{-\pi}^{\pi} \sin mx \sin nx\,\mathrm{d}x \begin{array}{l} = 0,\ m \neq n \\ = \pi,\ m = n \end{array}$$

Proof. Easy verification. ■

In terms of inner product these can be written as $\langle \sin mx,\ \sin nx\rangle = \langle \cos mx,\ \cos nx\rangle = 0$ if $m \neq n$ and $\langle \sin nx,\ \sin nx\rangle = \langle \cos nx,\ \cos nx\rangle = \pi$, also $\langle \sin mx,\ \cos nx\rangle = \langle 1,\ \sin nx\rangle = \langle 1,\ \cos nx\rangle = 0$ for all $m, n \in \mathbb{N}$.

DEFINITION. 3 A trigonometric series of the form $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, where the coefficients a_n, b_n are independent of x, is called a *Fourier Series*, the coefficients $a_0, a_n, b_n, n \in \mathbb{N}$, are called the *Fourier coefficients* of the series.

DEFINITION. 4 For an integrable function $f : [-\pi, \pi] \to \mathbb{R}$, the series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is called the *Fourier Series of the function* f where the Fourier coefficients are defined by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x, n = 0, 1, 2, \ldots \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x, n = 1, 2, \ldots,$$

and it is written as $f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.

It can be noted that in terms of the inner product defined above, $a_n = \frac{1}{\pi}\langle f(x),\ \cos nx\rangle$ and $b_n = \frac{1}{\pi}\langle f(x),\ \sin nx\rangle$.

THEOREM. 5 *If the series* $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ *converges uniformly to a function* f *in the interval* $[-\pi, \pi]$ *then the Fourier series of* f *is* $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ *where* $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x,\ n = 0, 1, 2, \ldots$ *and* $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x,\ n = 1, 2, \ldots$.

PROOF. Given that

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1}$$

Since the convergence is uniform, term by term integration is possible. So,

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x &= \int_{-\pi}^{\pi}\frac{1}{2}a_0\,\mathrm{d}x + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\,\mathrm{d}x \\ &= \frac{1}{2}a_0\int_{-\pi}^{\pi}\mathrm{d}x + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\,\mathrm{d}x + b_n\int_{-\pi}^{\pi}\sin nx\,\mathrm{d}x\right] \\ &= \pi a_0 \text{ (by using Theorem 1)}\end{aligned}$$

Hence $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$.

Let k be a positive integer, multiplying both sides of (1) by $\cos kx$ the series becomes

$$f(x)\cos kx = \frac{1}{2}a_0\cos kx + \sum_{n=1}^{\infty}(a_n \cos nx \cos kx + b_n \sin nx \cos kx). \tag{2}$$

It can be shown that this series also is uniformly convergent and hence term by term integration is allowed. Integrating both sides in the interval $[-\pi, \pi]$ we have

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos kx \, dx &= \int_{-\pi}^{\pi} \frac{1}{2}a_0 \cos kx \, dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\cos kx \, dx \\ &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos kx \, dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx \cos kx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos kx \, dx\right) \\ &= a_k \int_{-\pi}^{\pi} \cos kx \cos kx \, dx = \pi a_k \text{ (using Theorems 1 and 2).}\end{aligned}$$

Hence $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx \, dx$ for all $k \in \mathbb{N}$. Similarly, multiplying both sides of (1) by $\sin kx$ and integrating over $[-\pi, \pi]$, we obtain that $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx \, dx$, for all $k \in \mathbb{N}$. Hence the Fourier series of f is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. ■

REMARK. 6 It is established that if the series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges uniformly to a function f then the Fourier series of f is the given series, i.e. f is equal to its Fourier series. But for an arbitrary integrable function f the Fourier series of f may not be convergent in $[-\pi, \pi]$. Even, if converges, the sum function may not be be equal to f in $[-\pi, \pi]$.

DEFINITION. 7 A function $f : \mathbb{R} \to \mathbb{R}$ is called a *periodic function* if there exists $p \in R$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. The smallest positive number p satisfying the relation $f(x + p) = f(x)$ for all x in $\mathbb{R}$ is called the *period* of f.

For any $n \in \mathbb{N}$ the functions $\sin nx, \cos nx$ are periodic functions having the period $\frac{2\pi}{n}$ and, in particular, the functions $\sin x, \cos x$ are of period 2π. Since the Fourier series of a function is a linear combination of $\sin nx, \cos nx$, the sum function must be periodic of period 2π.

EXAMPLE. 8 Find the Fourier series for the function $f(x) = x, -\pi \le x \le \pi$.

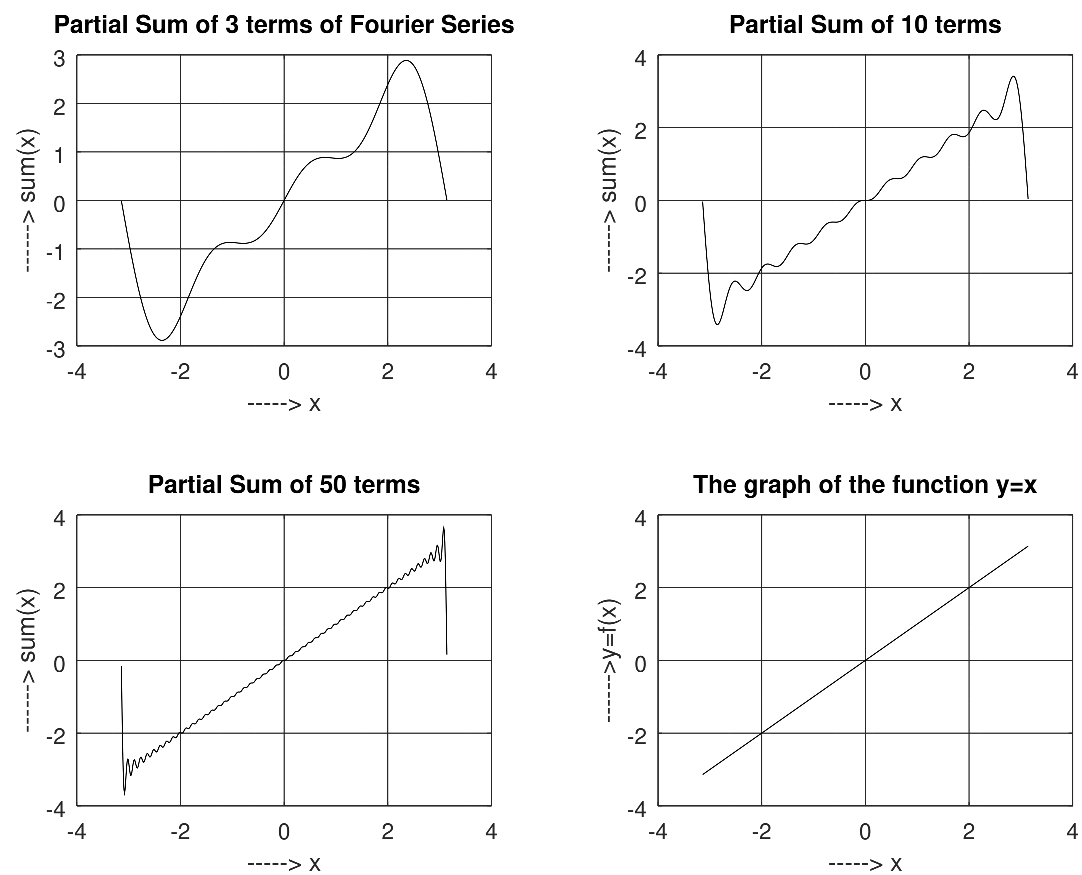
Let $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ be the Fourier series of the given function. Then

$$\begin{aligned}a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \, dx = 0 \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx \, dx = 0 \text{ (since } x\cos nx \text{ is an odd function)}\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,\mathrm{d}x \;=\; \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left[x\int \sin nx\,\mathrm{d}x - \int (1\int \sin nx\,\mathrm{d}x)\,\mathrm{d}x\right]_0^{\pi} \;=\; \frac{2}{\pi}\left[x\frac{-\cos nx}{n} + \int \frac{\cos nx}{n}\,\mathrm{d}x\right]_0^{\pi} \\
&= \frac{2}{\pi}\left(\frac{-\pi\cos n\pi}{n} + \left[\frac{\sin n\pi}{n^2}\right]_0^{\pi}\right) \;=\; \frac{2(-1)^{n+1}}{n}.
\end{aligned}$$

Hence the Fourier series of the function f is $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx$, i.e., $f(x) \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx$.

Below we draw the graph of the function $y = x$ along with the partial sum of the series for n terms, taking $n = 3, 10$ and $n = 50$.



It is clear from the graph that the series converges to f everywhere in $(-\pi, \pi)$ but does not converge at the endpoints $x = -\pi$ and $x = \pi$.

It has already been mentioned that as each of the functions $\sin nx, \cos nx$ are periodic of period $\frac{2\pi}{n}$ and hence repeats their values in the interval 2π, the sum of the Fourier series must be periodic of period 2π. If $f : [-\pi, \pi] \to \mathbb{R}$ is a function we can extend it over whole of $\mathbb{R}$.

To make f periodic we shall have $f(-\pi) = f(\pi)$ and hence we take the domain of f a half open interval, either $(-\pi, \pi]$ or $[-\pi, \pi)$. We take the former one, i.e., $f : (-\pi, \pi] \to \mathbb{R}$. Since $\mathbb{R} = \cup\{((2k-1)\pi, (2k+1)\pi] : k \in \mathbb{Z}\}$, for $x \in ((2k-1)\pi, (2k+1)\pi]$, $x - 2k\pi \in (-\pi, \pi]$. We extend f by, $f(x) = f(x - 2k\pi)$ for all $x \in ((2k-1)\pi, (2k+1)\pi]$, $k \in \mathbb{Z}$. This makes f periodic over $\mathbb{R}$.

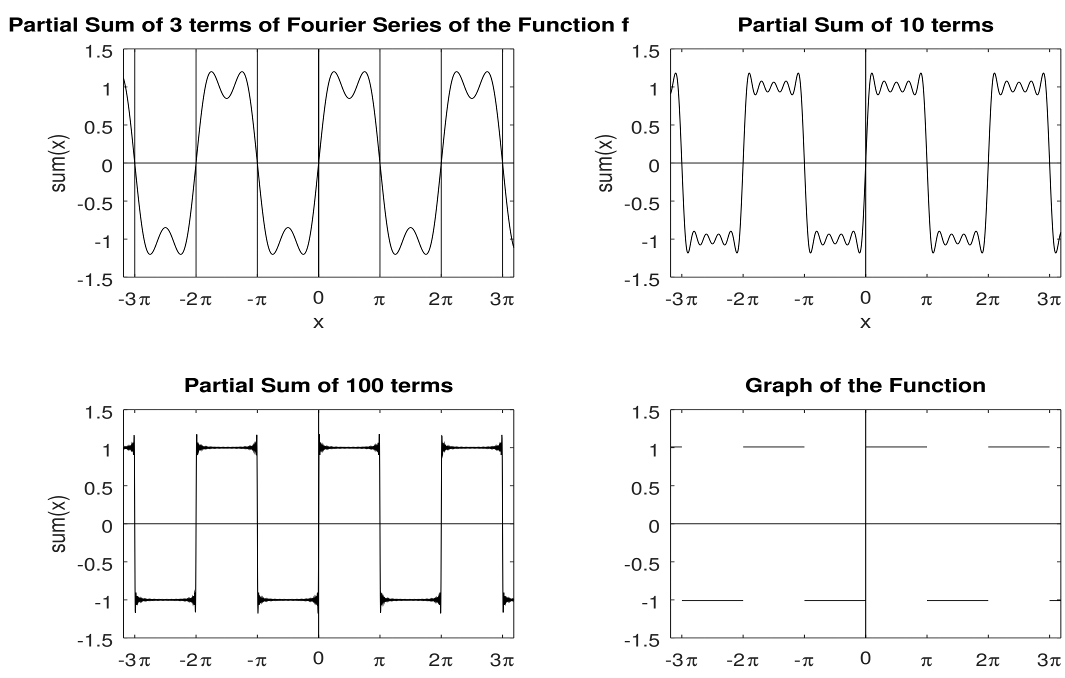
Below we give an example of a curve defined on $(-\pi, \pi]$ and its Fourier series extended over $\mathbb{R}$.

EXAMPLE. 9 The Fourier Series of the function $f(x) = -1, -\pi < x < 0, f(x) = 1, 0 \leq x \leq \pi$ extended periodically over $\mathbb{R}$.

As before, if $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the Fourier series, then

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-1)\,\mathrm{d}x + \int_{0}^{\pi}(1)\,\mathrm{d}x\right) = 0 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-\cos nx)\,\mathrm{d}x + \int_{0}^{\pi}\cos nx\,\mathrm{d}x\right) = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-\sin nx)\,\mathrm{d}x + \int_{0}^{\pi}\sin nx\,\mathrm{d}x\right) \\
&= \frac{2 - 2\cos n\pi}{n\pi} = \frac{2}{n\pi}(1 - (-1)^n).
\end{aligned}
$$

Hence $f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - (-1)^n)\sin nx$.



The graph of the partial sum of the series for n terms, taking $n = 3, 10$ and $n = 100$ is given above.

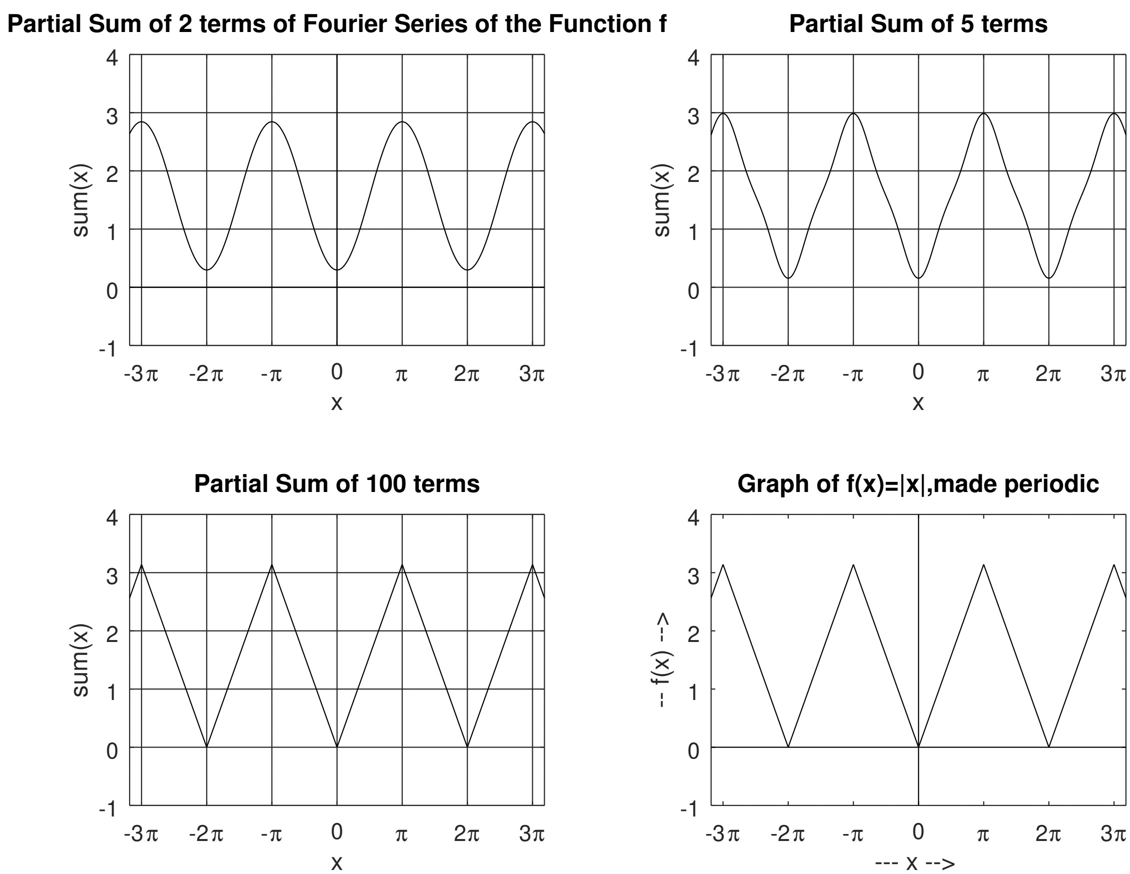
Here, also, we see that the series converges everywhere in $\mathbb{R}$ except at the points $x = k\pi, k \in \mathbb{Z}$. Note that the function is discontinuous at these points but the partial sums of the series is continuous.

EXAMPLE. 10 The Fourier Series of the function $f(x) = |x|, -\pi < x \leq \pi$ extended periodically over $\mathbb{R}$.

The function $f(x) = |x|$ is defined by $f(x) = -x$, if $x < 0$ and $f(x) = x$ if $x \geq 0$, which is an even function in $[-\pi, \pi]$. So $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\,\mathrm{d}x = \pi$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\cos nx = \frac{2}{\pi n^2}((-1)^n - 1)$ for all $n \in \mathbb{N}$. Also $b_n = \int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = 0$ for all $n \in \mathbb{N}$.

Hence the Fourier series for $f(x) = |x|$ is $\frac{1}{2}\pi + \sum\limits_{n=1}^{\infty} \frac{2}{\pi n^2}((-1)^n - 1)\cos nx$.

The graph of the partial sums for $n = 2, n = 5$ and $n = 100$ along with the graph of the function extended periodically over $\mathbb{R}$ is shown below.



It is clear from the graph that the sum of the series converges to f everywhere.

PROBLEM. 11 Find the Fourier series of the following functions:

1. $f(x) = |\cos x|, -\pi \le x \le \pi$.
2. $$\begin{aligned} f(x) &= 0, \ -\pi \le x < 0 \\ &= x, \ 0 \le x \le \pi. \end{aligned}$$
3. $f(x) = x^3, -\pi \le x \le \pi$.
4. $f(x) = e^{2x}, -\pi \le x \le \pi$.
5. $f(x) = x \sin x, -\pi \le x \le \pi$.
6. $f(x) = x^3, -\pi \le x \le \pi$.
7. $f(x) = x + \sin x, -\pi \le x \le \pi$.
8. $f(x) = x + x^2, -\pi \le x \le \pi$.

Fourier Series of odd and even functions

We know that if $f : [-a, a] \to \mathbb{R}$ is an even function, i.e., $f(-x) = f(x)$ for all $x \in [-a, a]$, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$ and if $f : [-a, a] \to \mathbb{R}$ is an odd function, i.e., $f(-x) = -f(x)$ for all $x \in [-a, a]$, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$, provided f is integrable. Also it is known that $\cos x$ is an even function and $\sin x$ is an odd function. It immediately follows that the product of two odd functions or the product of two even functions is an even function and the product of an even function and an odd function is an odd function.

Assume that $f : [-\pi, \pi] \to \mathbb{R}$ is an odd function. Then $f(x)\cos nx$ is an odd function and $f(x)\sin nx$ is an even function. Hence $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = 0$ and $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = 0$ for all $n \in \mathbb{N}$. Also $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} f(x)\sin nx\,\mathrm{d}x$ for all $n \in \mathbb{N}$. Hence the Fourier Series of an odd function f we have $a_n = 0$ for all $n \ge 0$ and its Fourier Series becomes $\sum\limits_{n=1}^{\infty} b_n \sin nx$, which we call *a series of sine functions* or briefly *a sine series.*

In a similar manner we can show that for an even function f defined on $[-\pi, \pi]$, $b_n = 0$ for all $n \in \mathbb{N}$ and hence its Fourier series becomes $\frac{1}{2}a_0 + \sum\limits_{n=1}^{\infty} a_n \cos nx$ which is called *a series of cosine functions* or briefly a *cosine series.*

In the examples 8 and 9 the functions were odd functions whereas in example 10 the function was $|x|$ which is an even function.

Half-Range Series

When a function f is defined in the interval $[0, \pi]$ it can be extended over $[-\pi, \pi]$ and hence its Fourier series can be determined. To do this the value of f to be defined for x in $[-\pi, 0)$.

Usually the following three ways are followed:

1. (i) Define $f(x) = f(-x)$ for all $x \in [-\pi, 0)$, then f becomes an even function in $[-\pi, \pi]$ and its Fourier series is a cosine series.

2. (ii) Define $f(x) = -f(-x)$ for all $x \in [-\pi, 0)$, then f becomes an odd function in $[-\pi, \pi]$ and hence its Fourier series is a sine series.

3. (iii) Define $f(x) = 0$ for all $x \in [-\pi, 0)$. Then f becomes neither odd nor even in $[-\pi, \pi]$ unless f is identically zero in that interval.

EXAMPLE. 12 Consider the function $f(x) = x^2, 0 \leq x \leq \pi$. We shall extend f in $[\pi, \pi]$ in the three ways stated above.

1. Define $f(x) = x^2$ for all $x \in [-\pi, \pi]$. Then f being even function $b_n = 0$ for all n in $\mathbb{N}$.

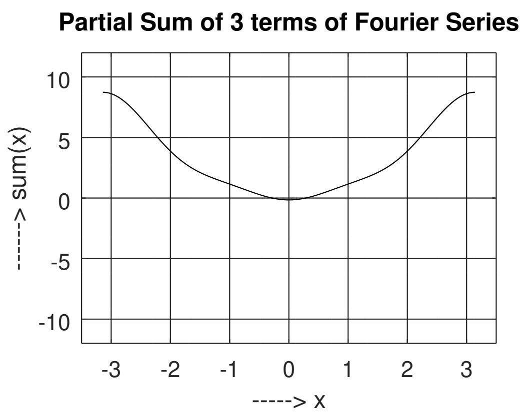
$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}x \;=\; 2\frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \;=\; \frac{2}{3}\pi^2. \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,\mathrm{d}x \;=\; \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left[x^2\int\cos nx\,\mathrm{d}x - \int 2x\frac{\sin nx}{n}\,\mathrm{d}x\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n}\right]_0^{\pi} - \frac{4}{n\pi}\left[\int x\sin nx\,\mathrm{d}x\right]_0^{\pi} \\
&= 0 - \frac{4}{n\pi}\left[x\frac{-\cos nx}{n} - \int 1\cdot\frac{\cos nx}{n}\,\mathrm{d}x\right]_0^{\pi} \;=\; \frac{4}{n^2}\cos n\pi = (-1)^n\frac{4}{n^2}.
\end{aligned}
$$

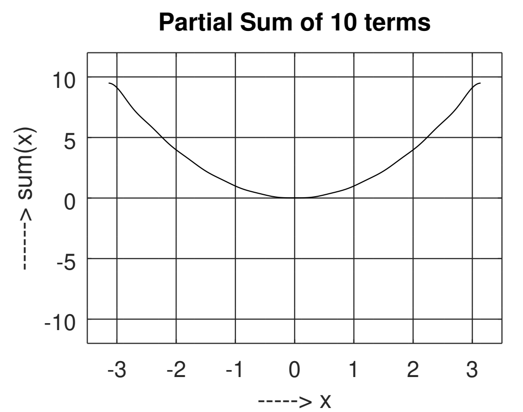
Hence the Fourier series of the function becomes

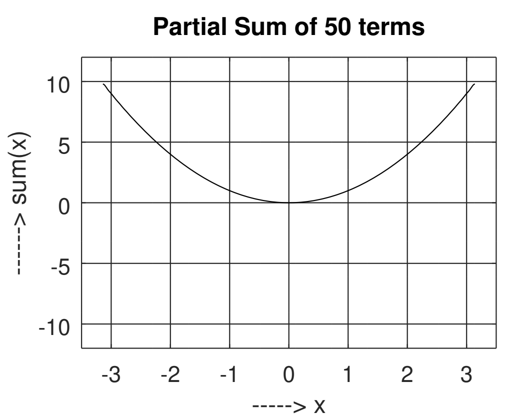
$$
\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx \;=\; \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx,
$$

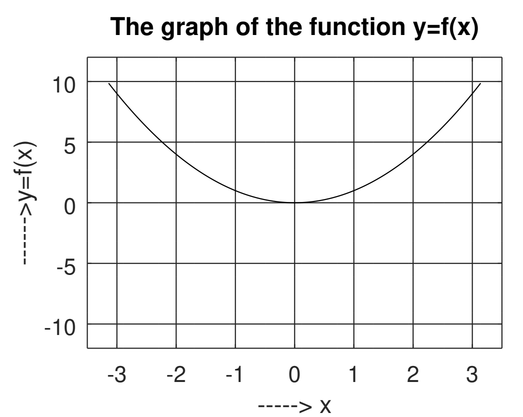
which is a cosine series.

The graph of the partial sums of the Fourier series and the function is given here.







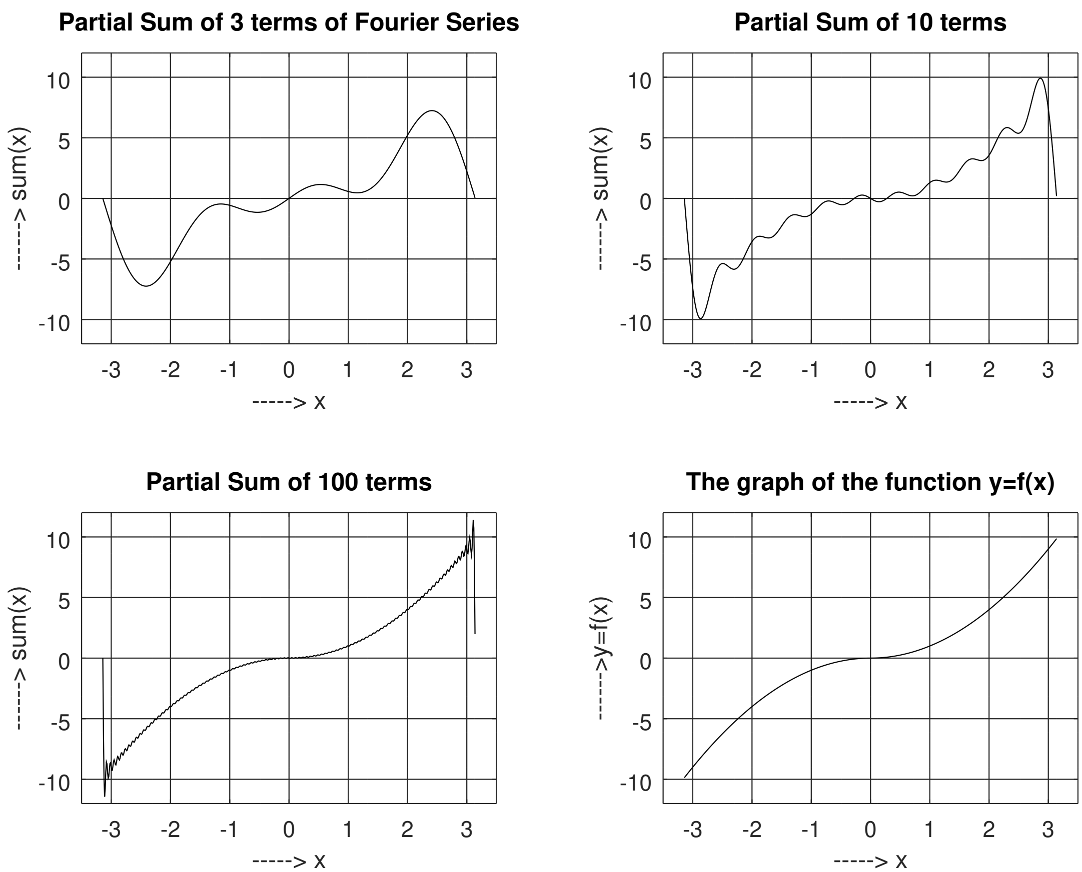


2. Define $f(x) = -x^2$ for all $x \in [-\pi, 0]$ and $f(x) = x^2$ for all $x \in [0, \pi]$. Then f is an odd function and hence $a_n = 0$ for all $n \in \mathbb{N}$.

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \;=\; \frac{1}{\pi}\int_{-\pi}^{0}(-x^2)\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x^2\sin nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,\mathrm{d}x \qquad \text{(putting } x=-y \text{ in the first integral)} \\
&= \frac{2}{\pi}\left[x^2\int \sin nx\,\mathrm{d}x - \int 2x\frac{-\cos nx}{n}\,\mathrm{d}x\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[x^2\frac{-\cos nx}{n}\right]_0^{\pi} + \frac{4}{n\pi}\left[\int x\cos nx\,\mathrm{d}x\right]_0^{\pi} \\
&= \frac{2\pi^2}{n\pi}(-1)^{n+1} + \frac{4}{n\pi}\left[x\frac{\sin nx}{n} - \int 1\frac{\sin nx}{n}\,\mathrm{d}x\right]_0^{\pi} \\
&= \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}((-1)^n - 1).
\end{aligned}
$$

Hence $f(x) \sim \sum_{n=1}^{\infty}[\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}((-1)^n - 1)]\sin nx,$ which is a sine series.

The graph of the partial sums of the Fourier series and the function is given here.



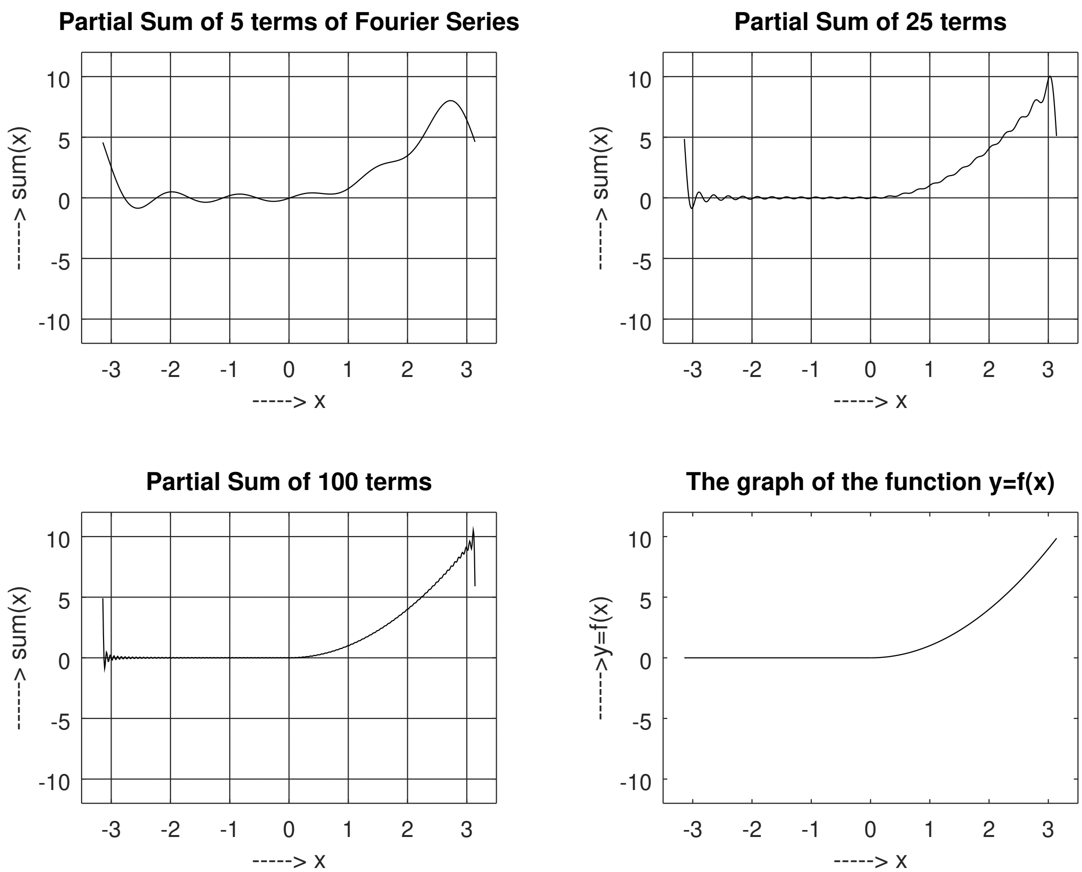
3. In this case define $f(x) = 0$ for $-\pi \le x < 0$, and $f(x) = x^2$ for $0 \le x \le \pi$. f is neither even nor odd.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \;=\; \frac{1}{\pi}\int_{0}^{\pi} x^2\,\mathrm{d}x \;=\; \frac{\pi^2}{3}.\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \;=\; \frac{1}{\pi}\int_{0}^{\pi} x^2\cos nx\,\mathrm{d}x\\
&= (-1)^n\frac{2}{n^2}.\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \;=\; \frac{1}{\pi}\int_{0}^{\pi} x^2\sin nx\,\mathrm{d}x\\
&= \frac{\pi}{n}(-1)^{n+1} + \frac{2}{n^3\pi}((-1)^n - 1).
\end{aligned}
$$

Hence the Fourier series is

$$
\frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[(-1)^n\frac{2}{n^2}\cos nx + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2}{n^3\pi}((-1)^n - 1)\right)\sin nx\right].
$$

The graph of the partial sums of the Fourier series for $n = 5, n = 25$ and $n = 100$ and the graph of the function are given below.



PROBLEM. 13 Find the cosine series of the following functions:

1. $f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$
2. $f(x) = \sin x,\ 0 \leq x \leq \pi.$
3. $f(x) = x,\ 0 \leq x \leq \pi.$
4. $f(x) = x^3,\ 0 \leq x \leq \pi.$

PROBLEM. 14 Find the sine series of the following functions:

1. $f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$
2. $f(x) = \cos x,\ 0 \leq x \leq \pi.$
3. $f(x) = x,\ 0 \leq x \leq \pi.$
4. $f(x) = x^3,\ 0 \leq x \leq \pi.$

Fourier Series in an arbitrary interval

Let a function f be defined and integrable in an interval $[-L, L]$. Consider a transformation $x = h(y) = \frac{Ly}{\pi}, -\pi \leq y \leq \pi$. Then $g = f \circ h$ is defined on $[-\pi, \pi]$ and integrable there. The Fourier series of g is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos ny + b_n \sin ny)$ where $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} g(y) \cos ny \, dy$ and $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} g(y) \sin ny \, dy$. We can revert from the variable y to x by the formula $y = \frac{\pi x}{L}$. Then $g(y) = g(\frac{\pi x}{L}) = f(x)$ and when $y = \pi, x = L$, when $y = -\pi, x = -L$, also $dy = \frac{\pi}{L} dx$. Hence the Fourier coefficients become

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-L}^{L} f(x) \cos(\tfrac{n\pi x}{L}) \tfrac{\pi}{L} \, dx = \frac{1}{L}\int_{-L}^{L} f(x) \cos \tfrac{n\pi x}{L} \, dx \\
b_n &= \frac{1}{\pi}\int_{-L}^{L} f(x) \sin(\tfrac{n\pi x}{L}) \tfrac{\pi}{L} \, dx = \frac{1}{L}\int_{-L}^{L} f(x) \sin \tfrac{n\pi x}{L} \, dx.
\end{aligned}
$$

The Fourier series of f becomes

$$
f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos \tfrac{n\pi x}{L} + b_n \sin \tfrac{n\pi x}{L}), x \in [-L, L].
$$

Example. 15 Fourier series of the function $f : [-2, 2] \to \mathbb{R}$ defined by,

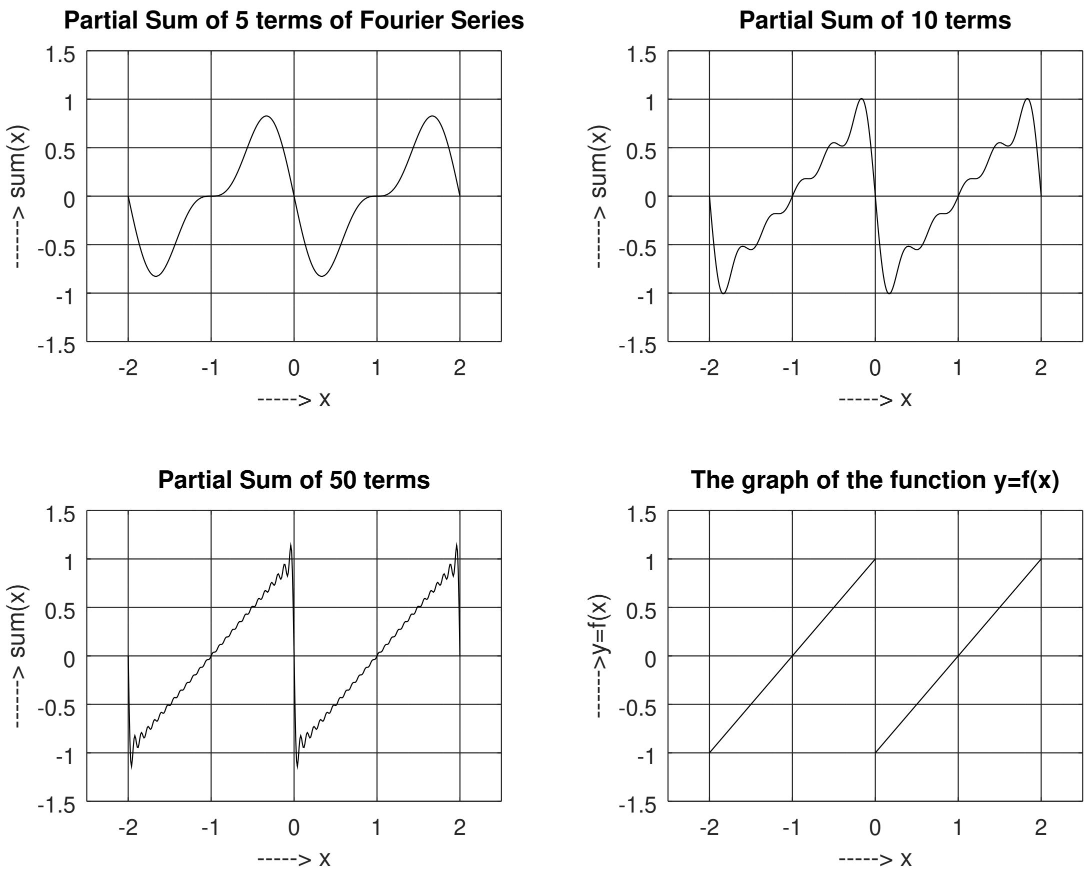
$$
f(x) = \begin{cases} x + 1, & -2 \leq x < 0 \\ x - 1, & 0 \leq x \leq 2. \end{cases}
$$

The function is an odd function and hence $a_n = 0$ for all $n \geq 0$.

$$
\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x) \sin \frac{n\pi x}{2} \, dx = \int_{0}^{2} (x-1) \sin \frac{n\pi x}{2} \, dx \\
&= \left[(x-1)\int \sin \frac{n\pi x}{2} \, dx - \int 1 \cdot \int \sin \frac{n\pi x}{2} \, dx \, dx\right]_0^2 \\
&= \frac{2}{n\pi}\left[(x-1)(-1)\cos \frac{n\pi x}{2} + \int \cos \frac{n\pi x}{2} \, dx\right]_0^2 \\
&= \frac{2}{n\pi}\left[(x-1)(-1)\cos \frac{n\pi x}{2} + \frac{2}{n\pi}\sin \frac{n\pi x}{2}\right]_0^2 = \frac{2}{n\pi}\left[-\cos n\pi - 1\right] \\
&= \frac{2}{n\pi}\left[(-1)^{n+1} - 1\right].
\end{aligned}
$$

Hence $f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[(-1)^{n+1} - 1\right] \sin(\frac{n\pi x}{2})$, $x \in [-2, 2]$.

The graph of the partial sums of the Fourier series of the function is given below.



Problem. 16 Find the Fourier series of the following functions in the intervals stated:

1. $f(x) = \begin{cases} 1, & -2 \le x < 0 \\ -1, & 0 \le x \le 2. \end{cases}$
2. $f(x) = \begin{cases} 0, & -2 \le x < 0 \\ x, & 0 \le x \le 2. \end{cases}$
3. $f(x) = x^2, -1 \le x \le 1.$
4. $f(x) = 1 - |x|, -1 \le x \le 1.$
5. $f(x) = x + x^2, -1 \le x \le 1.$

Properties of Fourier Series

Definition. 17 Let $\frac{1}{2}a_0 + \sum(a_n \cos nx + b_n \sin nx)$ be a Fourier series. For $n \in \mathbb{N}$ the partial sum $\frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$ is called the n-th partial sum and is denoted by S_n.

It can be noted that the n-th partial sum is actually a sum of $(2n+1)$ terms.

THEOREM. 18 (Bessel's Inequality) *If* $f : [-\pi, \pi] \to \mathbb{R}$ *is integrable function and* a_n, b_n *are the Fourier coefficients of* f *then* $\frac{1}{2}a_0^2 + \sum\limits_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x$.

PROOF. For $n \in \mathbb{N}$ define $S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), x \in [-\pi, \pi]$. Then

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x) - S_n(x))^2\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[(f(x))^2 - 2f(x)S_n(x) + (S_n(x))^2\right]\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x - \frac{2}{\pi}\int_{-\pi}^{\pi}f(x)S_n(x)\,\mathrm{d}x + \frac{1}{\pi}\int_{-\pi}^{\pi}(S_n(x))^2\,\mathrm{d}x. \qquad (1)\end{aligned}$$

Now,

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)S_n(x)\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\left(\frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\right)\,\mathrm{d}x \\ &= \frac{a_0}{2}\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,\mathrm{d}x + \sum_{k=1}^{n}\left(\frac{a_k}{\pi}\int_{-\pi}^{\pi}f(x)\cos kx\,\mathrm{d}x + \frac{b_k}{\pi}\int_{-\pi}^{\pi}f(x)\sin kx\,\mathrm{d}x\right) \\ &= \frac{1}{2}a_0^2 + \sum_{k=1}^{n}\left(a_k^2 + b_k^2\right). \qquad (2)\end{aligned}$$

Also

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}(S_n(x))^2\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\right)^2\,\mathrm{d}x \\ &= \frac{a_0^2}{4\pi}\int_{-\pi}^{\pi}\mathrm{d}x + \frac{1}{\pi}\sum_{k=1}^{n}\left(a_k^2\int_{-\pi}^{\pi}\cos^2 kx\,\mathrm{d}x + b_k^2\int_{-\pi}^{\pi}\sin^2 kx\,\mathrm{d}x\right) \\ &\qquad \text{(other integrals will vanish due to orthogonality relation)} \\ &= \frac{a_0^2}{4\pi}2\pi + \frac{1}{\pi}\sum_{k=1}^{n}\left(a_k^2\pi + b_k^2\pi\right) = \frac{1}{2}a_0^2 + \sum_{k=1}^{n}\left(a_k^2 + b_k^2\right). \qquad (3)\end{aligned}$$

Since $\displaystyle\int_{-\pi}^{\pi}(f(x) - S_n(x))^2\,\mathrm{d}x \geq 0$ for all $n \in \mathbb{N}$, using (1), (2) and (3),

$$\begin{aligned}0 &\leq \int_{-\pi}^{\pi}(f(x) - S_n(x))^2\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x - 2\left(\frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right) + \left(\frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x - \left(\frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right),\end{aligned}$$

which implies that $\frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2) \;\le\; \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x$. Since this is true for all $n \in \mathbb{N}$ it follows that $\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \;\le\; \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x$. ■

COROLLARY. 19 *For an integrable function f defined on $[-\pi, \pi]$, the series $\sum a_n^2$ and $\sum b_n^2$ are convergent and hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.*

PROOF. Sine $\sum a_n^2 \le \frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x < \infty$, it follows that $\sum a_n^2$ is bounded above and hence is convergent. Similarly $\sum b_n$ is also convergent. This imply that $\lim a_n^2 = 0$ and $\lim b_n^2 = 0$ and hence $\lim a_n = \lim b_n = 0$. ■

THEOREM. 20 (Parseval's Identity) *If f is integrable in $[-\pi, \pi]$, a_n, b_n are the Fourier coefficients of f then the identity, called the Parseval's Identity,*

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x$$

is true if and only if $\lim_{n\to\infty}\int_{-\pi}^{\pi}(f(x) - S_n(x))^2\,\mathrm{d}x = 0$.

PROOF. From the proof of Theorem 18 it follows that

$$\int_{-\pi}^{\pi}(f(x) - S_n(x))^2\,\mathrm{d}x \;=\; \int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x - \left[\frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right].$$

Since this is true for every $n \in \mathbb{N}$ we can take limit as $n \to \infty$, hence,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}(f(x) - S_n(x))^2\,\mathrm{d}x \;=\; \int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x - \left[\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)\right].$$

Hence $\int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x - \left[\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)\right] = 0$ if and only if $\lim_{n\to\infty}\int_{-\pi}^{\pi}(f(x) - S_n(x))^2\,\mathrm{d}x = 0$, i.e., $\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \int_{-\pi}^{\pi}(f(x))^2\,\mathrm{d}x$ if and only if $\lim_{n\to\infty}\int_{-\pi}^{\pi}(f(x) - S_n(x))^2\,\mathrm{d}x = 0$. ■

THEOREM. 21 (Reimann Lebesgue lemma) *For a function f integrable in an interval $[a,\ b]$,*

$$\lim_{n\to\infty}\int_a^b f(x)\cos nx\,\mathrm{d}x = 0 \;\textit{and}\; \lim_{n\to\infty}\int_a^b f(x)\sin nx\,\mathrm{d}x = 0.$$

PROOF. Since the interval $[a, b]$ is arbitrary, we consider the following cases:

Case I: When $[a, b] \subset [-\pi, \pi]$, we define a function $g : [-\pi, \pi] \to \mathbb{R}$ by $g(x) = f(x)$ when $x \in [a, b]$ and $g(x) = 0$ when $x \in [-\pi, \pi]\setminus[a, b]$. Then $\int_a^b f(x)\cos nx\,\mathrm{d}x = \int_{-\pi}^{\pi} g(x)\cos nx\,\mathrm{d}x =$

$\pi a_n, n \geq 0$. Hence by Corollary 19, $\lim\limits_{n\to\infty} \int_a^b f(x)\cos nx\,\mathrm{d}x = \lim\limits_{n\to\infty} \int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \lim\limits_{n\to\infty} \pi a_n = 0$.

Similarly, $\lim\limits_{n\to\infty} \int_a^b f(x)\sin nx\,\mathrm{d}x = 0$.

Case II: When $[a, b] \subset [(k-1)\pi, (k+1)\pi]$ for some integer k. Then extend f to the entire $[(k-1)\pi, (k+1)\pi]$ by taking $f(x) = 0$ for $x \in [(k-1)\pi, (k+1)\pi] \setminus [a, b]$. Define $g : [-\pi, \pi] \to \mathbb{R}$ by $g(x) = f(x + k\pi), -\pi \leq x \leq \pi$. Then, since the functions $\sin x, \cos x$ are periodic of period 2π, we have $\displaystyle\int_a^b f(x)\cos nx\,\mathrm{d}x = \int_{(k-1)\pi}^{(k+1)\pi} f(x)\cos nx\,\mathrm{d}x = \int_{-\pi}^{\pi} g(x)\cos nx\,\mathrm{d}x = \pi a_n$. Similarly, $\displaystyle\int_a^b f(x)\sin nx\,\mathrm{d}x = \pi b_n$. Hence by using Corollary 19 the result follows.

Case III: When $[a, b]$ does not contained in $[(k-1)\pi, (k+1)\pi]$ for any integer k. Then we divide the interval $[a, b]$ into subintervals $[a, c_1), [c_1, c_2), \ldots, [c_k, b]$ such that each of these subintervals is contained in $[(k-1)\pi, (k+1)\pi]$ for some integer k. Then,

$$\int_a^b f(x)\cos nx = \int_a^{c_1} f(x)\cos nx + \int_{c_1}^{c_2} f(x)\cos nx + \cdots + \int_{c_k}^{b} f(x)\cos nx.$$

Taking limit as $n \to \infty$, by Case II each of the integrals in right hand side vanish. Hence

$$\lim_{n\to\infty} \int_a^b f(x)\cos nx = 0.$$

Similar for the other integral. ■

Corollary. 22 *If f is integrable in $[-\pi, \pi]$ and $-\pi \leq a < b \leq \pi$ then,*

$$\lim_{n\to\infty} \int_a^b f(x)\sin(n + \tfrac{1}{2})x\,\mathrm{d}x = 0.$$

Proof. Since $\sin(n+\frac{1}{2})x = \sin nx \cos\frac{1}{2}x + \cos nx \sin\frac{1}{2}x$, we have $\lim\limits_{n\to\infty} \int_a^b f(x)\sin(n+\frac{1}{2})x\,\mathrm{d}x = \lim\limits_{n\to\infty} \int_a^b [f(x)\sin\frac{1}{2}x]\cos nx\,\mathrm{d}x + \lim\limits_{n\to\infty} \int_a^b [f(x)\cos\frac{1}{2}x]\sin nx\,\mathrm{d}x$.

Since $f(x)\sin\frac{1}{2}x$ and $f(x)\cos\frac{1}{2}x$ are integrable functions, by Riemann Lebesgue Lemma we have $\lim\limits_{n\to\infty} \int_a^b [f(x)\sin\frac{1}{2}x]\cos nx\,\mathrm{d}x = 0$ and $\lim\limits_{n\to\infty} \int_a^b [f(x)\cos\frac{1}{2}x]\sin nx\,\mathrm{d}x = 0$. Hence the result follows. ■

We observed in examples that the Fourier series of a function f converges to f at some points of the domain of f and does not converge at some other points. This naturally leads to the question – at which points (of the domain of f) the series converges to f, and at which points the series does not converge to f, or whether it converges at all? To answer the above question we should first define *Dirichlet Kernel* which will occur frequently in what follows.

DEFINITION. 23 For $n \in \mathbb{N}$ the function $D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}$ for $x \neq 2k\pi$ and $D_n(2k\pi) = n + \frac{1}{2}$, $x \in \mathbb{R}$, and $k \in \mathbb{Z}$, is called the *Dirichlet Kernel.*

Since $\lim\limits_{x\to 0} \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x} = \lim\limits_{x\to 0} \frac{\cos(n+\frac{1}{2})x\times(n+\frac{1}{2})}{2\cos\frac{1}{2}x\times\frac{1}{2}} = n + \frac{1}{2}$, $D_n(x)$ is continuous everywhere.

LEMMA. 24 *The function $D_n(x)$ has the following properties:* (i) *D_n is an even function,* (ii) *D_n is a periodic function of period 2π and* (iii) $\int_{-\pi}^{\pi} D_n(x)\,\mathrm{d}x = \pi$.

PROOF. For $k \in \mathbb{N}$ we have $\sin(k+\frac{1}{2})x - \sin(k-\frac{1}{2})x = 2\cos kx \sin\frac{1}{2}x$. Hence,

$$\sum_{k=1}^{n} 2\cos kx \sin\tfrac{1}{2}x \;=\; \sum_{k=1}^{n}(\sin(k+\tfrac{1}{2})x - \sin(k-\tfrac{1}{2})x) \;=\; \sin(n+\tfrac{1}{2})x - \sin\tfrac{1}{2}x.$$

which implies that

$$\begin{aligned} \sin\tfrac{1}{2}x + \sum_{k=1}^{n} 2\cos kx \sin\tfrac{1}{2}x &= \sin(n+\tfrac{1}{2})x, \\ \text{i.e.,}\quad \sin\tfrac{1}{2}x\left(1 + \sum_{k=1}^{n} 2\cos kx\right) &= \sin(n+\tfrac{1}{2})x \\ \text{or,}\quad \frac{1}{2} + \sum_{k=1}^{n} \cos kx &= \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x} = D_n(x). \end{aligned}$$

Hence $D_n(x) = \frac{1}{2} + \sum\limits_{k=1}^{n} \cos kx$ which is an even function and also a periodic function of period 2π. Finally, $\int_{-\pi}^{\pi} D_n(x)\,\mathrm{d}x = \int_{-\pi}^{\pi}(\frac{1}{2} + \sum\limits_{k=1}^{n}\cos kx)\,\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi}\mathrm{d}x + \sum\limits_{k=1}^{n}\int_{-\pi}^{\pi}\cos kx\,\mathrm{d}x = \pi$. ■

LEMMA. 25 *For a periodic function f of period p, for any $a \in \mathbb{R}$,* $\displaystyle\int_0^p f(x)\,\mathrm{d}x = \int_a^{a+p} f(x)\,\mathrm{d}x$.

PROOF. Since $f(x+p) = f(x)$, $\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(x+p)\,\mathrm{d}x$, replacing $x+p = z$, $\int_0^a f(x+p)\,\mathrm{d}x = \int_p^{a+p} f(z)\;dz$. Hence,

$$\begin{aligned} \int_0^p f(x)\,\mathrm{d}x &= \int_0^a f(x)\,\mathrm{d}x + \int_a^p f(x)\,\mathrm{d}x = \int_p^{a+p} f(z)\;dz + \int_a^p f(x)\,\mathrm{d}x \\ &= \int_a^p f(x)\,\mathrm{d}x + \int_p^{a+p} f(x)\,\mathrm{d}x = \int_a^{a+p} f(x)\,\mathrm{d}x. \end{aligned}$$

This completes the proof. ■

The above lemma says that if f is a periodic function of period p then the integral of f over any interval of length p remains same.

THEOREM. 26 [Convolution Theorem:] *If f is integrable and periodic of period 2π then*

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-u)D_n(u)\ \mathrm{d}u$$

where S_n is the n-th partial sum of the Fourier series of f and D_n is the Dirichlet kernel.

PROOF. Putting the values of the Fourier coefficients a_k, b_k in the expression of S_n,

$$\begin{aligned} S_n(x) &= \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k\cos kx + b_k \sin kx) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\ \mathrm{d}t + \sum_{k=1}^{n}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\ \mathrm{d}t\cos kx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\ \mathrm{d}t\sin kx\right) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}(\cos kt\cos kx + \sin kt \sin kx)\right)\ \mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos k(x-t)\right)\ \mathrm{d}t \ = \ \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)\ \mathrm{d}t. \end{aligned}$$

Put $x - t = u$, then $\mathrm{d}t = -\ \mathrm{d}u$ and when $t = -\pi, u = x + \pi$ and when $t = \pi, u = x - \pi$. Hence,

$$\begin{aligned} S_n(x) &= \frac{1}{\pi}\int_{x+\pi}^{x-\pi} f(x-u)D_n(u)(-\ \mathrm{d}u) \ = \ \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(x-u)D_n(u)\ \mathrm{d}u \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-u)D_n(u)\ \mathrm{d}u \text{ (Since both } f \text{ and } D_n \text{ are periodic of period } 2\pi) \ \blacksquare. \end{aligned}$$

The relation *convolution of f with D_n* is defined as $(f * D_n)(x) = \int_{-\pi}^{\pi} f(t)D_n(x-t)\ \mathrm{d}t$. Hence from the proof of the above theorem it reveals that for a periodic function f of period 2π the n-th partial sum is given by $S_n(x) = (f * D_n)(x)$, $n \in \mathbb{N}$.

COROLLARY. 27 *If f is integrable and periodic of period 2π then*

$$S_n(x) = \frac{1}{\pi}\int_{0}^{\pi} [f(x+u) + f(x-u)]D_n(u)\ \mathrm{d}u$$

where S_n is the n-th partial sum of the Fourier series of f and D_n is the Dirichlet kernel.

PROOF. We have from convolution theorem

$$\begin{aligned} S_n(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-u)D_n(u)\ \mathrm{d}u \\ &= \frac{1}{\pi}\int_{-\pi}^{0} f(x-u)D_n(u)\ \mathrm{d}u + \frac{1}{\pi}\int_{0}^{\pi} f(x-u)D_n(u)\ \mathrm{d}u. \end{aligned}$$

Replacing $u = -v$ in the first integral, $\mathrm{d}u = -\,\mathrm{d}v$, when $u = -\pi, v = \pi$ and when $u = 0, v = 0$, and since D_n is an even function, we have

$$\begin{aligned} S_n(x) &= \frac{1}{\pi}\int_{\pi}^{0} f(x+v)D_n(-v)(-\,\mathrm{d}v) + \frac{1}{\pi}\int_0^{\pi} f(x-u)D_n(u)\,\mathrm{d}u \\ &= \frac{1}{\pi}\int_0^{\pi} f(x+v)D_n(v)\,\mathrm{d}v + \frac{1}{\pi}\int_0^{\pi} f(x-u)D_n(u)\,\mathrm{d}u \\ &= \frac{1}{\pi}\int_0^{\pi} f(x+u)D_n(u)\,\mathrm{d}u + \frac{1}{\pi}\int_0^{\pi} f(x-u)D_n(u)\,\mathrm{d}u. \\ &= \frac{1}{\pi}\int_0^{\pi} [f(x-u)+f(x+u)]D_n(u)\,\mathrm{d}u. \end{aligned}$$

Hence the result. ■

LEMMA. 28 *If f is integrable and periodic of period 2π then for any function $S(x)$ defined on $\mathbb{R}$,*

$$S_n(x) - S(x) = \frac{1}{\pi}\int_0^{\pi} [f(x+u)+f(x-u)-2S(x)]D_n(u)\,\mathrm{d}u.$$

where $S_n(x)$ is the partial sum of the Fourier series of f and D_n is the Dirichlet kernel.

PROOF. We have $S_n(x) = \frac{1}{\pi}\int_0^{\pi}[f(x-u)+f(x+u)]D_n(u)\,\mathrm{d}u$. It is also known that $\int_0^{\pi} D_n(u)\,\mathrm{d}u = \frac{\pi}{2}$ which implies that $\int_0^{\pi} 2S(x)D_n(u)\,\mathrm{d}u = \pi S(x)$. Hence,

$$\begin{aligned} S_n(x) - S(x) &= \frac{1}{\pi}\int_0^{\pi} [f(x-u)+f(x+u)]D_n(u)\,\mathrm{d}u - \frac{1}{\pi}\int_0^{\pi} 2S(x)D_n(u)\,\mathrm{d}u \\ &= \frac{1}{\pi}\int_0^{\pi} [f(x-u)+f(x+u)-2S(x)]D_n(u)\,\mathrm{d}u. \end{aligned}$$

Hence the result. ■

LEMMA. 29 *If f is piecewise continuous and periodic of period 2π then*

$$S_n(x) - \tfrac{1}{2}[f(x+)+f(x-)] = \frac{1}{\pi}\int_0^{\pi} [f(x+u)-f(x+)]D_n(u)\,\mathrm{d}u + \frac{1}{\pi}\int_0^{\pi} [f(x-u)-f(x-)]D_n(u)\,\mathrm{d}u$$

where $S_n(x)$ is the partial sum of the Fourier series of f and D_n is the Dirichlet kernel.

PROOF. The existence of $f(x+)$ and $f(x-)$ follow from piecewise continuity of f. Taking $S(x) = \frac{1}{2}(f(x+)+f(x-))$ in the above lemma the result follows. ■

It is to be noted that piecewise continuity of f is only a sufficient condition of existence of $f(x+)$ and $f(x-)$ is its domain.

THEOREM. 30 [Dirichlet's Condition:] *If f is piecewise smooth and periodic of period 2π then*

$$\begin{aligned}\lim_{n\to\infty} S_n(x) &= \frac{f(x+0)+f(x-0)}{2}, \qquad \text{if } x \in (-\pi,\pi)\\ &= \frac{f(\pi-0)+f(-\pi+0)}{2}, \qquad \text{if } x = \pm\pi.\end{aligned}$$

where $S_n(x)$ denotes the n-th partial sum of the Fourier series of f.

PROOF. In view of Lemma 29 we have for $n \in \mathbb{N}, x \in (-\pi,\pi)$

$$\begin{aligned}&S_n(x) - \frac{1}{2}[f(x+)+f(x-)]\\ &= \frac{1}{\pi}\int_0^\pi [f(x+u)-f(x+)]D_n(u)\,\mathrm{d}u + \frac{1}{\pi}\int_0^\pi [f(x-u)-f(x-)]D_n(u)\,\mathrm{d}u\\ &= \frac{1}{\pi}\int_0^\pi \frac{[f(x+u)-f(x+)]}{2\sin\frac{1}{2}u}\sin(n+\tfrac{1}{2})u\,\mathrm{d}u + \frac{1}{\pi}\int_0^\pi \frac{[f(x-u)-f(x-)]}{2\sin\frac{1}{2}u}\sin(n+\tfrac{1}{2})u\,\mathrm{d}u\end{aligned}$$

Now the functions $\frac{[f(x+u)-f(x+)]}{2\sin\frac{1}{2}u}$ and $\frac{[f(x-u)-f(x-)]}{2\sin\frac{1}{2}u}$ are piecewise continuous on $0 < u \le \pi$. To check their continuity at $u = 0$ we apply L'Hospital's Theorem to get $\lim_{u\to 0+} \frac{[f(x+u)-f(x+)]}{2\sin\frac{1}{2}u} = f'(x+0)$ and $\lim_{u\to 0+} \frac{[f(x-u)-f(x-)]}{2\sin\frac{1}{2}u} = -f'(x-0)$.

Hence the functions $\frac{[f(x+u)-f(x+)]}{2\sin\frac{1}{2}u}$ and $\frac{[f(x-u)-f(x-)]}{2\sin\frac{1}{2}u}$ are piecewise continuous on $0 \le u \le \pi$ and hence integrable there. By Corollary 22 of Riemann Lebesgue Theorem it follows that

$$\lim_{n\to\infty}\int_0^\pi \frac{[f(x+u)-f(x+)]}{2\sin\frac{1}{2}u}\sin(n+\tfrac{1}{2})u\,\mathrm{d}u = 0$$

$$\text{and} \quad \lim_{n\to\infty}\int_0^\pi \frac{[f(x-u)-f(x-)]}{2\sin\frac{1}{2}u}\sin(n+\tfrac{1}{2})u\,\mathrm{d}u = 0.$$

Hence $\lim_{n\to\infty} S_n(x) - \frac{1}{2}[f(x+)+f(x-)] = 0$, i.e., $\lim_{n\to\infty} S_n(x) = \frac{1}{2}[f(x+)+f(x-)]$.

When $x = \pi$, since f is periodic of period 2π, $f(\pi+u) = f(-\pi+u)$ and hence $f(\pi+0) = f(-\pi+0)$. Thus $\frac{1}{2}(f(\pi-0)+f(\pi+0)) = \frac{1}{2}(f(\pi-0)+f(-\pi+0))$. Hence $\lim_{n\to\infty} S_n(\pi) = \frac{1}{2}(f(\pi-0)+f(-\pi+0))$. Similarly $\lim_{n\to\infty} S_n(-\pi) = \frac{1}{2}(f(-\pi-0)+f(-\pi+0)) = \frac{1}{2}(f(\pi-0)+f(-\pi+0))$.

This completes the proof.

EXAMPLE. 31 Find the Fourier serier series of the function $f(x) = x + x^2$, $-\pi < x \le \pi$ and hence deduce that $\frac{\pi^2}{6} = \sum \frac{1}{n^2}$.

If $\frac{1}{2}a_0 + \sum(a_n\cos nx + b_n\sin nx)$ is the Fourier series of f then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,\mathrm{d}x = \frac{2\pi^2}{3}$.

So $\frac{1}{2}a_0 = \frac{1}{3}\pi^2$. For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi}x\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,\mathrm{d}x \\ &= 0 + \frac{2}{\pi}\int_{0}^{\pi}x^2\cos nx\,\mathrm{d}x = (-1)^n\frac{4}{n^2}. \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nx\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi}x\sin nx\,\mathrm{d}x + 0 = (-1)^{n+1}\frac{2}{n}. \end{aligned}$$

Hence the Fourier series is $\frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}[(-1)^n\frac{4}{n^2}\cos nx + (-1)^{n+1}\frac{2}{n}\sin nx]$

The function $f(x) = x + x^2$ made periodic over $\mathbb{R}$ and hence is piecewise smooth. We can apply Dirichlet's criterion. The sum of the series at $x = \pi$ is

$$\frac{1}{2}(f(\pi-0) + f(-\pi+0)) = \frac{1}{2}(\pi^2 + \pi + \pi^2 - \pi) = \pi^2.$$

Also at $x = \pi$ the series becomes

$$\frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}[(-1)^n\frac{4}{n^2}\cos n\pi + (-1)^{n+1}\frac{2}{n}\sin n\pi] = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}(-1)^n = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}.$$

Hence we have $\pi^2 = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}$, i.e., $\frac{1}{6}\pi^2 = \sum_{n=1}^{\infty}\frac{1}{n^2}$.

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