Study Material on Improper Integral

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Improper Integral

We have already studied the theory of integration, where it was assumed that the following conditions are satisfied:

1. the integrand function is bounded over the interval of integration,
2. the interval of integration is bounded.

When any one or both of the above conditions are not satisfied, we still try to integrate the function by using the concept of limit. An integral of this type, when exists, is known as *improper integral.* The types of improper integrals can be classified as follows:

1. When the range of integration is finite but the integrand has an infinite discontinuity at any of the end points or in any interior point. This can be of the form:
   (a) $\int_a^b f(x)\,dx$ where f has an infinite discontinuity at $x = a$, for example, $\int_0^1 \frac{dx}{x^2}$.
   (b) $\int_a^b f(x)\,dx$ where f has an infinite discontinuity at $x = b$, for example, $\int_1^2 \frac{dx}{(x-2)^3}$.
   (c) $\int_a^b f(x)\,dx$ where f has an infinite discontinuity at $x = c$ where c is a point lying between a and b. For example, $\int_0^2 \frac{dx}{(x-1)^4}$.
   (d) $\int_a^b f(x)\,dx$ where f has a finite number of infinite discontinuities at $x = c_1, c_2, \ldots, c_k$ where $a \le c_1 < c_2 < \cdots < c_k \le b$. For example, $\int_0^{2\pi} \frac{\sin 2x}{\sin 2x - \cos 2x}dx$. Here the integrand has infinite discontinuities at $\pi/8, 5\pi/8, 9\pi/8$ and at $13\pi/8$.
2. When the range of integration is infinite, the integrand being a bounded function.
   (a) The integral of the type $\int_a^\infty f(x)\,dx$, where f is bounded in $[a, \infty)$.
   (b) The integral of the type $\int_{-\infty}^b f(x)\,dx$, where f is bounded in $(-\infty, b]$.
   (c) The integral of the type $\int_{-\infty}^\infty f(x)\,dx$, where f is bounded in $(-\infty, \infty)$.
3. The combination of the above two types is also possible, i.e., when the range of integration is infinite and also the integrand is an unbounded function.

When range of integration is finite:

We consider the improper integral $\int_a^b f(x)\,dx$ when $f(x)$ has a point of infinite discontinuity only at $x=a$. We take the integral $\int_{a+\epsilon}^b f(x)\,dx$ where $0<\epsilon<b-a$. This is a proper integral (as there is no other point of infinite discontinuity of f in this range). Suppose that $\int_{a+\epsilon}^b f(x)\,dx$ exists and equal to $\phi(\epsilon)$. If $\lim\limits_{\epsilon\to 0+0}\phi(\epsilon)$ exists and is finite, say equal to I, we say the improper integral $\int_a^b f(x)\,dx$ exists or converges at $x=a$ and write as

$$\int_a^b f(x)\,dx \;=\; \lim_{\epsilon\to 0+}\int_{a+\epsilon}^b f(x)\,dx \;=\; I.$$

Again, if $x=b$ be the only point of infinite discontinuity of f in the finite integral $[a,b]$, then $\int_a^b f(x)\,dx$ exists or converges at $x=b$ if $\lim\limits_{\epsilon\to 0+}\int_a^{b-\epsilon} f(x)\,dx$ exists, $0<\epsilon<b-a$. We then write,

$$\int_a^b f(x)\,dx \;=\; \lim_{\epsilon\to 0+}\int_a^{b-\epsilon} f(x)\,dx.$$

EXAMPLE. 1 $\int_0^1 \frac{dx}{x^2}$. Here $f(x)=\frac{1}{x^2}$ has only one point of infinite discontinuity at $x=0$. Then,

$$\int_0^1 \frac{dx}{x^2} \;=\; \lim_{\epsilon\to 0+}\int_\epsilon^1 \frac{dx}{x^2} \;=\; \lim_{\epsilon\to 0+}\left[-\frac{1}{x}\right]_\epsilon^1 \;=\; \lim_{\epsilon\to 0+}\left\{\frac{1}{\epsilon}-1\right\} \;=\; \infty.$$

Thus the integral $\int_0^1 \frac{dx}{x^2}$ does not converge.

EXAMPLE. 2 $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. Here $f(x)=\frac{1}{\sqrt{1-x^2}}$ has an infinite discontinuity at $x=1$. Then

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x^2}} \;&=\; \lim_{\epsilon\to 0+}\int_0^{1-\epsilon}\frac{dx}{\sqrt{1-x^2}} \;=\; \lim_{\epsilon\to 0+}\left[\,\sin^{-1}x\,\right]_0^{1-\epsilon}\\ &=\; \lim_{\epsilon\to 0+}\left\{\sin^{-1}(1-\epsilon)-\sin^{-1}0\right\} \;=\; \sin^{-1}1=\frac{\pi}{2}.\end{aligned}$$

So, the integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is convergent.

When both of a and b are the only points of infinite discontinuity of f in the finite range $[a,b]$, we take any point c where $a<c<b$ and consider the integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$. The integral $\int_a^b f(x)\,dx$ converges if the integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge at $x=a$ and $x=b$ respectively.
It can easily be verified that the result is independent of the choice of the point $x=c$. It requires just an adjustment by a proper integral.

Example. 3 $\displaystyle\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$. Here $f(x) = \frac{1}{\sqrt{x(1-x)}}$, both of $x = 0$ and $x = 1$ are the points of infinite discontinuity of f. We take $c = \frac{1}{2}$ and consider the integrals $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}}$ and $\displaystyle\int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{x(1-x)}}$. Now,

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} &= \lim_{\epsilon\to 0+} \int_\epsilon^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} = \lim_{\epsilon\to 0+} \int_\epsilon^{\frac{1}{2}} \frac{dx}{\sqrt{x-x^2)}} \\
&= \lim_{\epsilon\to 0+} \int_\epsilon^{\frac{1}{2}} \frac{dx}{\sqrt{(\frac{1}{2})^2 - (x-\frac{1}{2})^2}} = \lim_{\epsilon\to 0+} \left[\ \sin^{-1}(2x-1)\ \right]_\epsilon^{\frac{1}{2}} \\
&= \lim_{\epsilon\to 0+} \left\{\sin^{-1} 0 - \sin^{-1}(2\epsilon - 1)\right\} = -\sin^{-1}(-1) = \sin^{-1} 1 = \frac{\pi}{2}.
\end{aligned}$$

Thus the integral $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}}$ converges to $\frac{\pi}{2}$.

Also,

$$\begin{aligned}
\int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{x(1-x)}} &= \lim_{\epsilon\to 0+} \int_{\frac{1}{2}}^{1-\epsilon} \frac{dx}{\sqrt{x(1-x)}} = \lim_{\epsilon\to 0+} \int_{\frac{1}{2}}^{1-\epsilon} \frac{dx}{\sqrt{x-x^2)}} \\
&= \lim_{\epsilon\to 0+} \int_{\frac{1}{2}}^{1-\epsilon} \frac{dx}{\sqrt{(\frac{1}{2})^2 - (x-\frac{1}{2})^2}} = \lim_{\epsilon\to 0+} \left[\ \sin^{-1}(2x-1)\ \right]_{\frac{1}{2}}^{1-\epsilon} \\
&= \lim_{\epsilon\to 0+} \left\{\sin^{-1}(1-2\epsilon) - \sin^{-1} 0\right\} = \sin^{-1} 1 = \frac{\pi}{2}.
\end{aligned}$$

Thus the integral $\displaystyle\int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{x(1-x)}}$ converges to $\frac{\pi}{2}$. Hence, $\displaystyle\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ converges and

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} + \int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

When range of integration is infinite:

Consider the integral $\displaystyle\int_a^\infty f(x)\,dx$. Let f be bounded and integrable over $[a, X]$ where $X \geq a$. Then $\displaystyle\int_a^X f(x)\,dx$ exists and equal to, say, $\phi(X)$. If the limit $\displaystyle\lim_{X\to\infty} \phi(X)$ exists and finite, say I, then we say that the improper integral $\displaystyle\int_a^\infty f(x)\,dx$ converges with value I. Therefore,

$$\int_a^\infty f(x)\,dx = \lim_{X\to\infty} \int_a^X f(x)\,dx = I.$$

Consider the integral $\displaystyle\int_{-\infty}^b f(x)\,dx$. Let f be bounded and integrable over $[X, b]$ where $X \leq b$. If the limit $\displaystyle\lim_{X\to-\infty} \int_X^b f(x)\,dx$ exists and has finite value we say that the integral $\displaystyle\int_{-\infty}^b f(x)\,dx$ converges. We write,

$$\int_{-\infty}^b f(x)\,dx = \lim_{X\to-\infty} \int_X^b f(x)\,dx$$

Consider the integral $\int_{-\infty}^{\infty} f(x)\,dx$. Take any number c and consider the integrals $\int_{-\infty}^{c} f(x)\,dx$ and $\int_{c}^{\infty} f(x)\,dx$. If both the integrals Converge, then we say that $\int_{-\infty}^{\infty} f(x)\,dx$ converges and write,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx \\ &= \lim_{X\to-\infty}\int_{X}^{c} f(x)\,dx + \lim_{X'\to\infty}\int_{c}^{X'} f(x)\,dx.\end{aligned}$$

It is to be noted that the result is independent of the choice of c.

EXAMPLE. 4 Evaluate $\int_0^\infty \frac{dx}{1+x^2}$.

$$\begin{aligned}\int_0^\infty \frac{dx}{1+x^2} &= \lim_{X\to\infty}\int_0^X \frac{dx}{1+x^2} = \lim_{X\to\infty}\left[\ \tan^{-1} x\ \right]_0^X \\ &= \lim_{X\to\infty}\left[\tan^{-1} X - \tan^{-1} 0\right] = \lim_{X\to\infty} \tan^{-1} X = \frac{\pi}{2}\end{aligned}$$

PROBLEM. 5 Test the convergence of the following integrals and find the values when converge:

1. $\int_0^1 \frac{dx}{1-x}$
2. $\int_0^\infty \frac{x}{x^2+4}\,dx$
3. $\int_0^\infty \frac{dx}{x^2-1}$
4. $\int_0^\infty \frac{x\,dx}{(1+x^2)^2}$
5. $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}\,dx, a, b > 0$
6. $\int_{-\infty}^\infty \frac{x}{x^4+1}\,dx$
7. $\int_0^\infty \frac{dx}{x^2+2x+2}$
8. $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}, a, b > 0$

Tests for Convergence

We begin with a few useful examples.

EXAMPLE. 6 1. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if $n < 1$ and is divergent if $n \geq 1$.

The integral is proper if $n \leq 0$. for $n > 0$ and $n \neq 1$,

$$\begin{aligned}\int_a^b \frac{dx}{(x-a)^n} &= \lim_{\epsilon\to 0+}\int_{a+\epsilon}^b \frac{dx}{(x-a)^n} = \lim_{\epsilon\to 0+}\left[\frac{1}{-n+1}(x-a)^{-n+1}\right]_{a+\epsilon}^b \\ &= \frac{1}{1-n}\cdot\lim_{\epsilon\to 0+}\left[(b-a)^{1-n} - \epsilon^{1-n}\right].\end{aligned}$$

When $n > 1$ then $\lim_{\epsilon\to 0+} \epsilon^{1-n} = \infty$ and when $n < 1$ then $\lim_{\epsilon\to 0+} \epsilon^{1-n} = 0$. Thus,

$$\begin{aligned}\int_a^b \frac{dx}{(x-a)^n} &= \frac{1}{1-n}\cdot(b-a)^{1-n} \quad \text{if } n < 1 \\ &= \infty \text{ if } n > 1.\end{aligned}$$

For $n = 1$,

$$\int_a^b \frac{dx}{x-a} = \lim_{\epsilon\to 0+}\int_{a+\epsilon}^b \frac{dx}{x-a} = \lim_{\epsilon\to 0+}[\log(x-a)]_{a+\epsilon}^b = \lim_{\epsilon\to 0+}[\log(b-a)-\log\epsilon] = \infty.$$

Hence the integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if $n < 1$ and divergent if $n \geq 1$.

2. The integral $\int_a^\infty \frac{dx}{x^p}, a > 0$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

   For $p \neq 1$ the integral is evaluated as,

$$\begin{aligned}\int_a^\infty \frac{dx}{x^p} &= \lim_{X\to\infty}\int_a^X \frac{dx}{x^p} = \frac{1}{1-p}\cdot\lim_{X\to\infty}\left[x^{1-p}\right]_a^X = \frac{1}{1-p}\cdot\lim_{X\to\infty}[X^{1-p}-a^{1-p}]\\ &= \infty \ \text{ if } p < 1\\ &= \frac{1}{p-1}a^{1-p} \ \text{ if } p > 1.\end{aligned}$$

   For $p = 1$ the integral becomes,

$$\int_a^\infty \frac{dx}{x} = \lim_{X\to\infty}\int_a^X \frac{dx}{x} = \lim_{X\to\infty}[\log x]_a^X = \lim_{X\to\infty}[\log X - \log a] = \infty.$$

   Hence the integral is convergent if $p > 1$ and is divergent if $p \leq 1$.

Problem. 7 Test the convergence of the integral $\int_a^b \frac{dx}{(b-x)^n}$.

Theorem. 8 (Comparison test:) *If f and g are two non-negative functions defined on $(a, b]$, having the only infinite discontinuity at a and $f \leq g$ on $(a, c]$ for some $c, a < c \leq b$, then (i) if $\int_a^b g(x)\,dx$ is convergent then so is $\int_a^b f(x)\,dx$, (ii) if $\int_a^b f(x)\,dx$ is divergent then so is $\int_a^b g(x)\,dx$.*

Proof. Since $\lim_{\epsilon\to 0}\int_{a+\epsilon}^b f(x)\,dx = \lim_{\epsilon\to 0}\int_{a+\epsilon}^c f(x)\,dx + \int_c^b f(x)\,dx$, and the last integral is proper one, without any loss of generality we may assume $c = b$, i.e., $f \leq g$ on $(a, b]$. Then for any $\epsilon > 0$ since $0 \leq f(x) \leq g(x)$ for all $x \in [a+\epsilon, b]$, we have $\int_{a+\epsilon}^b f(x)\,dx \leq \int_{a+\epsilon}^b g(x)\,dx$.

Hence, (i) when $\int_a^b g(x)\,dx$ is convergent, then $\lim_{\epsilon\to 0+}\int_{a+\epsilon}^b f(x)\,dx \leq \lim_{\epsilon\to 0+}\int_{a+\epsilon}^b g(x)\,dx < \infty$. So $\int_{a+\epsilon}^b f(x)\,dx$ is convergent.

(ii) On the other hand, when $\int_a^b f(x)\,dx$ is divergent then $\lim_{\epsilon\to 0+}\int_{a+\epsilon}^b f(x)\,dx = \infty$. This implies that $\lim_{\epsilon\to 0+}\int_{a+\epsilon}^b g(x)\,dx = \infty$. Thus $\int_a^b g(x)\,dx$ is divergent. ■

In a similar way one can prove the following theorem and hence I omit it and ask the students to write the proof as an exercise.

THEOREM. 9 (Comparison test:) *If f, g are integrable over $[a, X)$ for all $X \geq a$ and $0 \leq f(x) \leq g(x)$ for all $x \in [a, \infty)$ then (i) if $\int_a^\infty g(x)\,dx$ is convergent then so is $\int_a^\infty f(x)\,dx$ and (ii) if $\int_a^\infty f(x)\,dx$ is divergent then so is $\int_a^\infty f(x)\,dx$.*

THEOREM. 10 *If the functions $f, g : (a, b] \to \mathbb{R}$ have the only point of infinite discontinuity at $x = a$, both f, g are positive on $(a, b]$ such that $\lim_{x \to a+} \frac{f(x)}{g(x)} = l$, where $0 < l < \infty$, then the integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either both converge or both diverge.*

PROOF. Since both f, g are positive, $l > 0$. Choose $\epsilon = \frac{l}{2}$. Then there exists $\delta > 0$ such that $a + \delta < b$ and $|\frac{f(x)}{g(x)} - l| < \epsilon$ for all $x \in (a, a + \delta)$, i.e., $l - \epsilon < \frac{f(x)}{g(x)} < l + \epsilon$ for all $x \in (a, a + \delta)$. Since $g > 0$ and $\epsilon = \frac{l}{2}$, we have

$$\frac{l}{2} \cdot g(x) < f(x) < \frac{3l}{2} \cdot g(x) \quad \text{for all} \quad x \in (a, a + \delta).$$

Put $c = a + \delta$. Assume that the integral $\int_a^b f(x)\,dx$ is convergent. This implies that

$$\begin{aligned}
\int_a^c f(x)\,dx \text{ is convergent} \quad &\Rightarrow \int_a^c \frac{l}{2}\, g(x)\,dx \text{ is convergent} \Rightarrow \int_a^c g(x)\,dx \text{ is convergent} \\
&\Rightarrow \int_a^c g(x)\,dx + \int_c^b g(x)\,dx \text{ is convergent (adding a proper integral)} \\
&\Rightarrow \int_a^b g(x)\,dx \text{ is convergent.}
\end{aligned}$$

Also assuming $\int_a^b f(x)\,dx$ is divergent, since $\int_c^b f(x)\,dx$ is a proper integral, we have

$$\begin{aligned}
\int_a^c f(x)\,dx \text{ is divergent} \quad &\Rightarrow \int_a^c \frac{3l}{2}\, g(x)\,dx \text{ is divergent} \Rightarrow \int_a^c g(x)\,dx \text{ is divergent} \\
&\Rightarrow \int_a^c g(x)\,dx + \int_c^b g(x)\,dx \text{ is divergent (adding a proper integral)} \\
&\Rightarrow \int_a^b g(x)\,dx \text{ is divergent.}
\end{aligned}$$

Hence either both the integrals are convergent or both are divergent. ■

The following theorem is stated without proof and the students are asked to write it by following the method adopted in the above one.

THEOREM. 11 *If the functions $f, g : [a, b) \to \mathbb{R}$ have the only point of infinite discontinuity at $x = b$, both f, g are positive on $[a, b)$ such that $\lim_{x \to b-} \frac{f(x)}{g(x)} = l$, where $0 < l < \infty$, then the integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either both converge or both diverge.*

Analogous results are valid for the integrals when the range of integration is infinite and the integrand has no infinite discontinuity.

THEOREM. 12 *Assume that $f, g : [a, \infty) \to \mathbb{R}$ are positive and has no infinite discontinuity on its domain, also $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$, where $0 < l < \infty$. Then the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are either both convergent or both divergent.*

PROOF. Since $l > 0$, choose $\epsilon > 0$ such that $l - \epsilon > 0$. For this ϵ there exists $m > a$ such that $|\frac{f(x)}{g(x)} - l| < \epsilon$ whenever $x > m$. This implies that $l - \epsilon < \frac{f(x)}{g(x)} < l + \epsilon$ whenever $x > m$, i.e., $(l-\epsilon)g(x) < f(x) < (l+\epsilon)g(x)$ whenever $x > m$. Since $\int_a^\infty f(x)\,dx = \int_a^m f(x)\,dx + \int_m^\infty f(x)\,dx$ and $\int_a^m f(x)\,dx$ is a proper integral, we have

$$\begin{aligned}\int_a^\infty f(x)\,dx \text{ is convergent} \quad &\Rightarrow \int_m^\infty f(x)\,dx \text{ is convergent} \Rightarrow \int_m^\infty (l-\epsilon)g(x)\,dx \text{ is convergent}\\ &\Rightarrow \int_m^\infty g(x)\,dx \text{ is convergent} \Rightarrow \int_a^\infty g(x)\,dx \text{ is convergent.}\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_a^\infty f(x)\,dx \text{ is divergent} \quad &\Rightarrow \int_m^\infty f(x)\,dx \text{ is divergent} \Rightarrow \int_m^\infty (l+\epsilon)g(x)\,dx \text{ is divergent}\\ &\Rightarrow \int_m^\infty g(x)\,dx \text{ is divergent} \Rightarrow \int_a^\infty g(x)\,dx \text{ is divergent.}\end{aligned}$$

Thus, either both are convergent or both are divergent. ∎

Analogously one can prove that

THEOREM. 13 *Assume that $f, g : (-\infty, b] \to \mathbb{R}$ are positive and has no infinite discontinuity on its domain, also $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = l$, where $0 < l < \infty$. Then the integrals $\int_{-\infty}^b f(x)\,dx$ and $\int_{-\infty}^b g(x)\,dx$ are either both convergent or both divergent.*

We omit its proof, interested students can do it as an exercise.

EXAMPLE. 14 Test the convergence of the integral $\displaystyle\int_0^1 \frac{dx}{x^{\frac{3}{2}}(1+x^2)^{\frac{5}{2}}}$.

Here $f(x) = \frac{1}{x^{\frac{3}{2}}(1+x^2)^{\frac{5}{2}}}$ has only infinite discontinuity at $x = 0$. Let us take $g(x) = \frac{1}{x^{\frac{3}{2}}}, 0 < x \le 1$. Then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{(1+x^2)^{\frac{5}{2}}} = 1 < \infty$. Hence both the integrals $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ have the same convergence behaviour. Since the integral $\int_0^1 \frac{dx}{x^{\frac{3}{2}}}$ is divergent, $(n = \frac{3}{2} > 1)$, the integral $\int_0^1 f(x)\,dx$ is divergent.

EXAMPLE. 15 Test the convergence of the integral $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x^2)^3}$.

Here $f(x) = \frac{x}{(1+x^2)^3}$ has no infinite discontinuity in $[0, \infty)$. Counting the degrees in the numerator and denominator of f we take $g(x) = \frac{1}{x^5}, x > 0$. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{(1+x^2)^3}\frac{x^5}{1} = \lim_{x\to\infty} \frac{x^6}{(1+x^2)^3} = \lim_{x\to\infty} \frac{1}{(\frac{1}{x^2}+1)^3} = 1 < \infty.$$

Since the integral $\int_0^\infty \frac{1}{x^5}\,dx$ is convergent $(p = 5 > 1)$ it follows that $\int_0^1 f(x)\,dx$ is convergent.

Beta and Gamma Function

In this section we deal with two improper integrals which have much importance in various applications of mathematics. They are known as Beta Functions and Gamma Functions.

Beta Function:

The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is known as *beta function* and is denoted by $\beta(m,n)$.

THEOREM. 16 *The beta function* $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ *is convergent when* $m > 0$ *and* $n > 0$.

PROOF. It is obvious that the integral is proper when both $m, n \geq 1$. So we have to check when $m < 1$ or $n < 1$ or both. We divide the integral as

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx + \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,dx.$$

When $m < 1$ the first integrand has an infinite discontinuity at $x = 0$ and when $n < 1$ the second integrand has an infinite discontinuity at $x = 1$. Let $f(x) = x^{m-1}(1-x)^{n-1}, 0 < x < 1$.

When $m < 1$: take $g(x) = x^{m-1}$. Then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{x^{m-1}(1-x)^{n-1}}{x^{m-1}} = \lim_{x\to 0+} (1-x)^{n-1} = 1$.

Now, $\int_0^{\frac{1}{2}} g(x)\,dx = \int_0^{\frac{1}{2}} x^{m-1}\,dx = \int_0^{\frac{1}{2}} \frac{1}{x^{1-m}}\,dx$ is convergent if $1 - m < 1$, i.e., if $m > 0$. Hence the integral $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx$ is convergent when $m > 0$.

When $n < 1$: take $h(x) = (1-x)^{n-1}$. Then $\lim_{x\to 1-} \frac{f(x)}{h(x)} = \lim_{x\to 1-} \frac{x^{m-1}(1-x)^{n-1}}{(1-x)^{n-1}} = \lim_{x\to 1-} x^{m-1} = 1$.

Now, $\int_{\frac{1}{2}}^1 h(x)\,dx = \int_{\frac{1}{2}}^1 (1-x)^{n-1}\,dx = \int_{\frac{1}{2}}^1 \frac{1}{(1-x)^{1-n}}\,dx$ is convergent if $1 - n < 1$, i.e., if $n > 0$. Hence the integral $\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent when $n > 0$.

From the above two, the integral $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent if $m > 0$ and $n > 0$. ■

Henceforth, whenever we write $\beta(m,n)$ we shall assume $m > 0$ and $n > 0$, unless stated otherwise.

Properties of Beta Functions

1. $\beta(m,n) = \beta(n,m)$.

   Putting $x = 1 - y$, $dx = -dy$, when $x \to 0, y \to 1$ and when $x \to 1, y \to 0$, we have,

$$\begin{aligned}\beta(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_1^0 (1-y)^{m-1}y^{n-1}(-dy)\\ &= \int_0^1 y^{n-1}(1-y)^{m-1}dy = \beta(n,m).\end{aligned}$$

2. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta.$

   Substituting $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\ d\theta$. When $x \to 0$ then $\theta \to 0$, when $x \to 1$, $\theta \to \pi/2$. So,

$$\begin{aligned}\beta(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta\ d\theta \\ &= 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta.\end{aligned}$$

3. $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx.$

   Take a substitution $x = \frac{y}{1+y}$. Then $dx = \frac{1}{(1+y)^2}\ dy$ and $1-x = \frac{1}{1+y}$. Also $x = \frac{y}{1+y}$ gives $y = \frac{x}{1-x}$, hence when $x \to 0, y \to 0$ and when $x \to 1, y \to \infty$. Thus the integral becomes,

$$\begin{aligned}\beta(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1}\left(\frac{1}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}\ dy \\ &= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\ dy.\end{aligned}$$

4. For $k > 0$, $\beta(m,n) = k\int_0^1 x^{mk-1}(1-x^k)^{n-1}\,dx.$

   This can be proved by substituting $x = y^k, k > 0$. The students are required to do it.

5. $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1).$

$$\begin{aligned}\beta(m,n+1) &= \int_0^1 x^{m-1}(1-x)^n\,dx = \int_0^1 x^{m-1}(1-x)^{n-1}(1-x)\,dx \\ &= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx - \int_0^1 x^m(1-x)^{n-1}\,dx \\ &= \beta(m,n) - \beta(m+1,n).\end{aligned}$$

   Hence the result.

EXAMPLE. 17 Evaluate $\beta(\frac{1}{2},\frac{1}{2})$.

$$\beta(\tfrac{1}{2},\tfrac{1}{2}) = 2\int_0^{\pi/2} \sin^{2\cdot\frac{1}{2}-1}\theta\ \cos^{2\cdot\frac{1}{2}-1}\theta\ d\theta = 2\int_0^{\pi/2} d\theta = 2\times\frac{\pi}{2} = \pi.$$

PROBLEM. 18 Prove that 1. $\beta(m,n+1) = \frac{m}{m+n}\cdot\beta(m,n)$ and 2. $\beta(m+1,n) = \frac{n}{m+n}\cdot\beta(m,n)$.

Gamma Function:

The improper integral $\int_0^\infty e^{-x}x^{n-1}\,dx$ is known as *Gamma Function* and is denoted by $\Gamma(n)$. Thus,

THEOREM. 19 *The Gamma Function* $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$ *is convergent when* $n > 0$.

PROOF. The function $f(x) = e^{-x}x^{n-1}$ has an infinite discontinuity at $x = 0$ when $n < 1$. So we have to check convergence both at $x = 0$ and at ∞. We write the integral as,

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx = \int_0^1 e^{-x}x^{n-1}\,dx + \int_1^\infty e^{-x}x^{n-1}\,dx.$$

To check convergence at 0, take $g(x) = x^{n-1}$. Then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{e^{-x}x^{n-1}}{x^{n-1}} = \lim_{x\to 0+} e^{-x} = 1 < \infty$. Also the integral $\int_0^1 g(x)\,dx = \int_0^1 x^{n-1}\,dx = \int_0^1 \frac{dx}{x^{1-n}}$ is convergent if $1 - n < 1$, i.e., if $n > 0$. Hence the integral $\int_0^1 e^{-x}x^{n-1}\,dx$ is convergent if $n > 0$.

We check the convergence at ∞ a little elaborately, in a few steps.

1. First note that $\int_1^\infty e^{-kx}\,dx$ is convergent for any $k > 0$. One can easily verify this from definition.
2. For any positive integer n, there exists $M > 0$ such that $e^{-x}x^{n-1} < e^{-\frac{1}{2}x}$ for all $x > M$. To verify this evaluate the limit $\lim_{x\to\infty} \frac{x^{n-1}}{e^{\frac{1}{2}x}} = 0$ [by using L' Hospital's rule ($\frac{\infty}{\infty}$) form]. Hence for taking $\epsilon = 1$ there exists $M > 0$ such that $|\frac{x^{n-1}}{e^{\frac{1}{2}x}} - 0| < 1$ whenever $x > M$, i.e., $x^{n-1} < e^{\frac{1}{2}x}$ for all $x > M$. Multiplying both sides by e^{-x} we have $e^{-x}x^{n-1} < e^{-\frac{1}{2}x}$ for all $x > M$. By comparison and using item 1 above, we have $\int_1^\infty e^{-x}x^{n-1}\,dx$ is convergent whenever n is a positive integer.
3. When n is a real number greater than 1 then $[n]$ is a positive integer and $e^{-x}x^{n-1} < e^{-x}x^{[n]}$. Since the integral $\int_1^\infty e^{-x}x^{[n]}\,dx$ is convergent, by comparison $\int_1^\infty e^{-x}x^{n-1}\,dx$ is convergent for any real number $n > 1$.
4. When $0 < n < 1$, since we have $1 \le x^{n-1} \le x$ for all $x > 1$, we get $\frac{1}{e^{\frac{1}{2}x}} \le \frac{x^{n-1}}{e^{\frac{1}{2}x}} \le \frac{x}{e^{\frac{1}{2}x}}$ for all $x > 1$. Since $\lim_{x\to\infty} \frac{1}{e^{\frac{1}{2}x}} = 0 = \lim_{x\to\infty} \frac{x}{e^{\frac{1}{2}x}}$, by sandwich rule we have $\lim_{x\to\infty} \frac{x^{n-1}}{e^{\frac{1}{2}x}} = 0$. By the method similar to that adopted in item 2 we can prove $\int_1^\infty e^{-x}x^{n-1}\,dx$ is convergent when $0 < n < 1$.

Hence the integral $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$ is convergent if $n > 0$. ■

Henceforth, whenever we write $\Gamma(n)$, we shall assume that $n > 0$, unless we state otherwise.

Properties of Gamma Function

1. $\Gamma(n+1) = n\Gamma(n)$.

We have
$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty e^{-x}x^n\,dx = \lim_{X\to\infty}\int_0^X e^{-x}x^n\,dx \\ &= \lim_{X\to\infty}\left\{\left[x^n(-e^{-x})\right]_0^X + n\int_0^X e^{-x}x^{n-1}\,dx\right\} \\ &= \lim_{X\to\infty}\left[-X^n e^{-X} + 0\right] + n\lim_{X\to\infty}\int_0^X e^{-x}x^{n-1}\,dx \\ &= \lim_{X\to\infty}\left[-\frac{X^n}{e^X}\right] + n\int_0^\infty e^{-x}x^{n-1}\,dx = 0 + n\Gamma(n) = n\Gamma(n).\end{aligned}$$

2. For a positive integer n, $\Gamma(n+1) = n!$.

$$\begin{aligned} \Gamma(n+1) &= n\Gamma(n) \\ \Gamma(n) &= (n-1)\Gamma(n-1) \\ \vdots &\quad \vdots \\ \Gamma(3) &= 2\Gamma(2) \\ \Gamma(2) &= 1\Gamma(1) \end{aligned}$$

Thus $\Gamma(n+1) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot \Gamma(1) = n!\cdot\Gamma(1)$. Since $\Gamma(1) = \int_0^\infty e^{-x}x^{1-1}\,dx = \lim_{X\to\infty}\int_0^X e^{-x}\,dx = \lim_{X\to\infty}\left[-e^{-x}\right]_0^X = \lim_{X\to\infty}\left[-\frac{1}{e^X}+1\right] = 1$, it follows that Thus $\Gamma(n+1) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$.

3. For $k>0$, $\Gamma(n) = k\int_0^\infty e^{-x^k}x^{kn-1}\,dx$.

Put $x = y^k, k>0$. Then when $x\to 0, y\to 0$ and when $x\to\infty, y\to\infty$. Also $dx = ky^{k-1}\,dy$. After this substitution the integral becomes,

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx = \int_0^\infty e^{-y^k}(y^k)^{n-1}ky^{k-1}\,dy = k\int_0^\infty e^{-y^k}y^{kn-1}\,dy.$$

Hence the result.

4. For $0<n<1$, $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$.

Proof of this result is omitted as it involves topics beyond the curriculum.

Relation between Beta Function and Gamma Function

THEOREM. 20 *For $m,n>0$, $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.*

PROOF. We know for $k>0$, $\Gamma(n) = k\int_0^\infty e^{-x^k}x^{kn-1}\,dx$. Hence taking $k=2$ we have

$$\begin{aligned} \Gamma(n)\Gamma(m) &= 2\int_0^\infty e^{-x^2}x^{2n-1}\,dx\cdot 2\int_0^\infty e^{-y^2}y^{2m-1}\,dy \\ &= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2n-1}y^{2m-1}\,dx\,dy \end{aligned}$$

Take $x = r\cos\theta$ and $y = r\sin\theta$, $0<r<\infty, 0\le\theta\le\frac{\pi}{2}$ and $\frac{\partial(x,y)}{\partial(r,\theta)} = r$, the integral becomes,

$$\begin{aligned} \Gamma(n)\Gamma(m) &= 4\int_{r=0}^\infty\int_{\theta=0}^{\frac{\pi}{2}} e^{-r^2}r^{2n-1}\cos^{2n-1}\theta\; r^{2m-1}\sin^{2m-1}\theta\; rd\theta dr \\ &= 4\int_{r=0}^\infty\int_{\theta=0}^{\frac{\pi}{2}} e^{-r^2}r^{2(n+m)-1}\cos^{2n-1}\theta\;\sin^{2m-1}\theta\; d\theta dr \\ &= 2\int_0^\infty e^{-r^2}r^{2(n+m)-1}\,dr\cdot 2\int_0^{\frac{\pi}{2}}\cos^{2n-1}\theta\;\sin^{2m-1}\theta\;d\theta \;=\; \Gamma(n+m)\cdot\beta(n,m). \end{aligned}$$

Hence $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Example. 21

1. Find $\Gamma(\frac{1}{2})$.

From the relation between beta and gamma functions it follows that $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left(\Gamma(\frac{1}{2})\right)^2}{\Gamma(1)} = \left(\Gamma(\frac{1}{2})\right)^2$. Since $\beta(\frac{1}{2}, \frac{1}{2}) = \pi$, therefore, $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi$. Hence $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

2. Establish the relation $\displaystyle\int_0^{\pi/2} \sin^p\theta\ \cos^q\theta\ d\theta = \frac{\beta(\frac{p+1}{2}, \frac{q+1}{2})}{2} = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})}$, $p, q > -1$.

$$\text{We know, } \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta.$$

Putting $2m-1 = p, 2n-1 = q$, $m = \frac{p+1}{2}, n = \frac{q+1}{2}$. When $m, n > 0$, then $p > -1, q > -1$. So we get

$$2\int_0^{\pi/2} \sin^p\theta\ \cos^q\theta\ d\theta = \beta(\frac{p+1}{2}, \frac{q+1}{2}),$$

Again, since $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$,

$$\beta(\frac{p+1}{2}, \frac{q+1}{2}) = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+1}{2} + \frac{q+1}{2})} = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q+2}{2})}.$$

Thus,

$$\int_0^{\pi/2} \sin^p\theta\ \cos^q\theta\ d\theta = \frac{1}{2}\cdot\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q+2}{2})}.$$

3. Evaluate $\displaystyle\int_0^{\pi/2} \sin^4\theta\ \cos^6\theta\ d\theta$.

$$\begin{aligned}\int_0^{\pi/2} \sin^4\theta\ \cos^6\theta\ d\theta &= \frac{\Gamma(\frac{4+1}{2})\ \Gamma(\frac{6+1}{2})}{2\Gamma(\frac{4+6+2}{2})} = \frac{\Gamma(\frac{5}{2})\ \Gamma(\frac{7}{2})}{2\Gamma(6)} = \frac{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})\ \frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{2\times 5!} \\ &= \frac{\frac{45}{32}\left[\Gamma(\frac{1}{2})\right]^2}{2\times 120} = \frac{45}{64\times 120}\pi = \frac{3}{64\times 8}\pi = \frac{3\pi}{512}.\end{aligned}$$

Problems

1. Prove that (i) $\beta(m,1) = \frac{1}{m}$, (ii) $\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$, (iii) $\Gamma(6) = 120$.
2. Evaluate the following integrals using beta and gamma functions:

(i) $\displaystyle\int_0^1 x^3(1-x^2)^{5/2}\,dx$ (ii) $\displaystyle\int_0^1 x^4(1-x^2)^3\,dx$ (iii) $\displaystyle\int_0^1 x^{3/2}(1-x)^{3/2}\,dx$

(iv) $\displaystyle\int_0^1 x^{5/2}(1-x)\,dx$ (v) $\displaystyle\int_0^{\pi/2} \cos^4 x\,dx$ (vi) $\displaystyle\int_0^{\pi/2} \sin^6 x\cos^3 x\,dx$

Ans: (i) $\frac{2}{63}$ (ii) $\frac{16}{1155}$ (iii) $\frac{3\pi}{128}$ (iv) $\frac{4}{63}$ (v) $\frac{3\pi}{16}$ (vi) $\frac{2}{63}$.