# Non-Homogeneous Linear Recurrence Relation with Constant coefficients

Suppose the $k^{th}$-order Non-Homogeneous Linear ~~eq~~ recurrence relation with constant coefficients is

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + c_2 a_{r-3} + \cdots + c_k a_{r-k} = f(r) \quad \cdots (1)$$

The solution of equation(1) is,

$$a_r = a_r^{(h)} + a_r^{(P)}$$

Where, $a_r^{(h)}$ = Homogeneous solution, $a_r^{(P)}$ = Particular Solution

Solution of Homogeneous Linear Recurrence relation when $f(r) = 0$.

## Method to find the Particular solution when $f(r) \neq 0$ :

To find the Particular solution of recurrence relation, we will consider a trial solution on the basis nature of $f(r)$.

Case-1 : When $f(r) = b^r$ and $b$ is not a root of characteristic equation

Suppose the trial solution is $a_r^{P} = A \cdot b^r$

Case 2 : When $f(r) = b^r$ and $b$ is a root of characteristic equation.

1. If multiplicity of $b$ is one i.e. $b$ is a root of multiplicity 1, then

$$a_r = A \cdot r \cdot b^r$$

2. If multiplicity of $b$ is two i.e. $b$ is a root of multiplicity 2, then

$$a_r = A \cdot r^2 \cdot b^r$$

3. If multiplicity of $b$ is three i.e. $b$ is a root of multiplicity 3, than

$$a_r = A \cdot r^3 b^r$$

## Example 1

Solve the recurrence relation $a_r - 5a_{r-1} + 6a_{r-2} = 5^r$

$\Rightarrow$ Given recurrence relation is

$$a_r - 5a_{r-1} + 6a_{r-2} = 5^r \quad \text{----} (1)$$

The Characteristic equation is

$$m^2 - 5m + 6 = 0$$
$$(m-2)(m-3) = 0$$
$$\therefore m = 2, 3$$

$\therefore$ The homogeneous solution is

$$a_r^{(h)} = c_1(2)^r + c_2(3)^r$$

Since $f(r) = 5^r$ and 5 is not root of homogeneous Solution, then we will assume the particular Solution is

$$a_r^{(P)} = A.(5)^r \quad \text{-----} (2)$$

Putting the value of $a_r$ in equation (1), we get

$$A.5^r - 5(A.5^{r-1}) + 6(A.5^{r-2}) = 5^r$$
$$\Rightarrow A.\cancel{5^r}\left[1 - 1 + \frac{6}{25}\right] = \cancel{5^r}$$
$$\Rightarrow A.\cancel{5^r}\left[\frac{6}{25}\right] = \cancel{5^r}\ 1$$
$$\Rightarrow A = 25/6$$

Putting in equation (2), we get

$$a_r^{(P)} = \frac{25}{6}(5)^r = \frac{1}{6}5^{r+2}$$

The total solution of equation (1) is given by —

$$a_r = a_r^{(h)} + a_r^{(P)}$$
$$\Rightarrow a_r = c_1(2)^r + c_2(3)^r + \frac{1}{6}5^{r+2}$$

## Problem - 2

Solve the recurrence relation $a_r - 7a_{r-1} + 10a_{r-2} = 3^r$, given that $a_0 = 0$ and $a_1 = 1$.

Solution: Given recurrence relation is

$$a_r - 7a_{r-1} + 10a_{r-2} = 3^r \quad \text{----} (1)$$

The characteristic equation is

$$m^2 - 7m + 10 = 0$$

$$\Rightarrow (m-2)(m-5) = 0$$

$$m = 2, 5$$

The homogeneous solution is

$$a_r^{(h)} = c_1(2)^r + c_2(5)^r$$

Since $f(r) = 3^r$ and 3 is not a root of homogeneous solution, then we will assume the particular solution is

$$a_r^{(P)} = A.(3)^r \quad \text{----} (2)$$

Putting the value of $a_r$ in equation (1), we get

$$A.3^r - 7(A.3^{r-1}) + 10(A.3^{r-2}) = 3^r$$

$$\Rightarrow A.3^r \left[1 - 7/3 + \frac{10}{9}\right] = 3^r$$

$$\Rightarrow A.3^r \left[-2/9\right] = 3^r$$

$$\Rightarrow A = -9/2$$

Putting in equation (2), we get

$$a_r^{(P)} = -9/2(3)^r = -\frac{1}{2}.3^{r+2}$$

The total Solution of equation(1) is given by

$$a_r = a_r^{(h)} + a_r^{(P)}$$

$$a_r = c_1(2)^r + c_2(5)^r - \frac{1}{2}3^{r+2} \quad \cdots (3)$$

Putting $a_0 = 0$ ie. $a_r = 0$ and $r = 0$ in equation (3), we get,

$$c_1(2)^0 + c_2(5)^0 - \frac{1}{2}\cdot 3^2 = 0$$

$$\Rightarrow c_1 + c_2 = 9/2 \quad \cdots\cdots (4)$$

Again Putting $a_1 = 1$ ie. $a_r = 1$ and $r = 1$ in equation (3), we get –

$$c_1(2) + c_2(5) - \frac{1}{2}\cdot 3^3 = 1$$

$$\Rightarrow 2c_1 + 5c_2 = 29/2 \quad \cdots\cdots (5)$$

Solving the equation (4) and (5), we get

$$c_1 = 8/3 \text{ and } c_2 = 11/6$$

Putting in equation (3), we get

$$a_r = \frac{8}{3}(2)^r + 11/6\,(5)^r - \frac{1}{2}\cdot 3^{r+2}$$

Case-2 Example



Solve the recurrence relation $a_r - 4a_{r-1} + 4a_{r-2} = 2^r$

Solution

Given recurrence relation is

$$a_r - 4a_{r-1} + 4a_{r-2} = 2^r \quad \text{- - - - (1)}$$

The characteristic equation is

$$m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)(m-2) = 0$$

$$m = 2, 2$$

The homogeneous solution is

$$a_r^{(h)} = (c_1 + c_2 r) 2^r$$

Since $f(r) = 2^r$ and 2 is ~~the~~ a characteristic root of multiplicity is 2, then we will assume the particular solution is -

$$a_r^{(p)} = A \cdot r^2 \cdot 2^r \quad \text{- - - - - (2)}$$

Putting the value of $a_r$ in equation (1), we get-

$$A \cdot r^2 \cdot 2^r - 4\left[A(r-1)^2 2^{r-1}\right] + 4\left[A \cdot (r-2)^2 \cdot 2^{r-2}\right] = 2^r$$

$$\Rightarrow A \cdot r^2 \cdot 2^r - 4\left[A \cdot (r^2 - 2r + 1) 2^{r-1}\right] + 4\left[A(r^2 - 4r + 4) 2^{r-2}\right] = 2^r$$

$$\Rightarrow \cancel{2^r} \cdot A\left[\cancel{r^2} - \cancel{2r^2} + \cancel{4r} - 2 + \cancel{r^2} - \cancel{4r} + 4\right] = \cancel{2^r}$$

$$\Rightarrow 2A = 1 \Rightarrow A = 1/2$$

Putting in equation (2) we get

$$a_r^{(p)} = \frac{1}{2} r^2 (2)^r = r^2 2^{r-1}$$

The total solution of equation (1) is given by —

$$\Rightarrow a_r = a_r^{(h)} + a_r^{(p)}$$

$$\Rightarrow a_r = (c_1 + c_2 r)(2)^r + r^2 \cdot 2^{r-1}$$

# Excercise

1) $a_r - 4a_{r-1} + 3a_{r-2} = 5^r$ [Answer: $a_r = c_1(1)^r + c_2(3)^r + \frac{5^{r}}{8}$]

2) $a_r - 2a_{r-1} + a_{r-2} = 2^r$ [Answer: $a_r = 1 - 2r + 2^r$]

3) $a_r = -5a_{r-1} - 6a_{r-2} + 42.4^r$, given by $a_2 = 278, a_3 = 962$

[Answer: $a_r = (-2)^r + 2.(-3)^r + 16.4^r$]

4) $a_r - 3a_{r-1} + 2a_{r-2} = 2^r$ [Answer: $a_r = c_1(1)^r + c_2(2)^r + r.2^{r+1}$]

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