Recurrence Relation:

A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, a_2, a_3, \cdots, a_{n-1}$, for all intergers n with $n \geq n_0$, where n_0 is non-negative integer.

Solution of recurrence relation:

A sequence is called a solution of a recurrence relation of its terms satisfy the recurrence relation.

Example: Let $\{a_n\}$ be a sequence that satisfies that recurrence relation $a_n = a_{n-1} - a_{n-2}$, for $n = 2, 3, \cdots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Solution

Given that $a_n = a_{n-1} - a_{n-2}$

Putting $n = 2, 3, \cdots$, so that

Therefore, $a_2 = a_1 - a_0 = 5 - 3 = 2$

$a_3 = a_2 - a_1 = 2 - 5 = -3$

Linear Recurrence Relation with constant cofficients

A recurrence relation of the form

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \cdots + c_k a_{r-k} = f(r) \cdots (1)$$

Where c_i's are constant, is called a Linear recurrence relation with constant coefficients.

The recurrence relation in equation (1) is known as a k^{th} order recurrence relation, provided that both $c_0 \neq 0$ and $c_k \neq 0$.

Second order Recurrence relation

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} = f(r)$$

Third order Recurrence relation

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + c_3 a_{r-3} = f(r)$$

The solution of equation (1) is,

$a_r = a_r^{(h)} + a_r^{(P)}$ where $a_r^{(h)}$ = Homogeneous solution
and, $a_r^{(P)}$ = Particular Solution

Homogeneous Recurrence Relation

If $f(r) = 0$, then equation (1) is called Homogeneous recurrence relation.

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + c_3 a_{r-3} + \cdots + c_k a_{r-k} = 0$$

Non-Homogeneous Recurrence Relation

If $f(r) \neq 0$, then equation (1) is called non-homogeneous recurrence relation.

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \cdots + c_k a_{r-k} = f(r).$$

Homogeneous Linear Recurrence Relation with constant coefficient

Suppose the second order Homogeneous Linear recurrence relation is

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} = 0$$

The characteristic Equation is

$$c_0 m^2 + c_1 m + c_2 = 0$$

Case 1 : If the roots of characteristic equation are real and unequal.

say, $m_1 \neq m_2$

$\therefore$ The general solution is

$$a_r = c_1 m_1^r + c_2 m_2^r$$

Case 2 : If the roots of characteristic equation are Real and equal.

say, $m_1 = m_2 = m$

$\therefore$ The general solution is

$$a_r = (c_1 + r c_2) m^r$$

Case 3 : If the roots of characteristic equation of are in complex numbers

say, $m = \alpha \pm i\beta$

$\therefore$ The general solution is

$$a_r = (c_1 \cos r\theta + c_2 \sin r\theta) R^r$$

where, $R = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}(\beta/\alpha)$.